이학박사 학위논문

On the stochastic regularity of diffusion processes associated with (non-symmetric) Dirichlet forms

((비대칭) 디리클레 형식과 연관된 확산 과정의 확률적 정칙성)

2015년 8월

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On the stochastic regularity of diffusion processes associated with (non-symmetric) Dirichlet forms

A dissertation
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Abstract

First for any starting point in $\mathbb{R}^d$ we identify the stochastic differential equation of distorted Brownian motion with respect to a certain discontinuous Muckenhoupt A_2-weight under the assumption of Fukushima's absolute continuity condition. We then systematically develop general tools to apply the absolute continuity condition. These tools comprise methods to obtain a Hunt process on a locally compact separable metric state space whose transition function has a density w.r.t. the reference measure and methods to estimate drift potentials comfortably. Our results are applied to distorted Brownian motions and construct weak solutions to singular stochastic differential equations, i.e. equations with possibly unbounded and discontinuous drift and reflection terms which may be the sum of countably many local times. The solutions can start from any point of the explicitly specified state space. We consider different kinds of weights, like Muckenhoupt A_2 weights and weights with moderate growth at singularities as well as different kind of (multiple) boundary conditions. We also apply the general schemes to degenerate elliptic forms and show solutions to the corresponding stochastic differential equations. Finally we extend the results of symmetric distorted Brownian motions to non-symmetric ones. Using elliptic regularity results in weighted spaces, stochastic calculus and the theory of non-symmetric Dirichlet forms, we first show weak existence of non-symmetric distorted Brownian motion for any starting point in some domain E of $\mathbb{R}^d$, where E is explicitly given as the points of strict positivity of the unique continuous version of the density to its invariant measure. Once having shown weak existence, we obtain from a result of [43] that the constructed weak solution is indeed strong as well as pathwise unique up to its explosion time. As a consequence of our approach, we can use the theory of Dirichlet forms to prove further properties of the solutions. More precisely, we obtain new non-explosion criteria for them.

Key words: (non-symmetric) distorted Brownian motion, diffusion processes, (non-symmetric) Dirichlet forms, strong existence, absolute continuity condition, Muckenhoupt weights
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Contents

Chapter 1

Introduction

1.1 Introduction

This thesis is based on the four papers [57, 58, 59, 55]. Given the (non-symmetric) Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E, \mu)$ where E is a locally compact separable metric space and μ is a well-chosen positive Radon measure, we are mainly concerned with pointwise solution to the corresponding stochastic differential equation associated with the Dirichlet form. In the case of symmetric regular Dirichlet form by Fukushima decomposition (see [35, Theorem 5.2.2]) we obtain for $u \in D(\mathcal{E})$

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^{[u]} + N_t^{[u]}, \quad \mathbb{P}_x\text{-a.s. for all } x \in E \setminus N, \tag{1.1}$$

where N is some exceptional (in general not explicitly known) set, $\tilde{u}$ is a quasi-continuous μ-version of $u \in D(\mathcal{E})$, $M_t^{[u]}$ is a martingale additive functional of finite energy, and $N_t^{[u]}$ is a continuous additive functional of zero energy. The Dirichlet form approach is first an L^2-approach. It then leads from an L^2-analysis to a quasi-sure analysis w.r.t. the corresponding capacity, similarly to the theory of weighted Sobolev spaces. Then the question is whether this energy method can be refined to a pointwise (stochastic) analysis, or at least to an analysis for specified points. The question whether the set N in (1.1) can be explicitly specified or chosen, for instance to be the empty set, can be regarded as a regularity theory for stochastic differential equations similarly to the regularity theory of partial differential equations in analysis. In order to explicitly specify N in (1.1) we apply the theory of strict Fukushima decomposition to a large class of various diffusion processes, i.e. distorted Brownian motion and diffusion processes associated with degenerate elliptic forms. In the case of non-symmetric distorted Brownian motion by using elliptic regularity and martingale problem we show pathwise unique strong solutions. Now we describe the results in more detail.

Distorted skew Brownian motion w.r.t. discontinuous Muckenhoupt weights

In Chapter 2 we are concerned with the construction of a weak solution to the singular stochastic differential equation

$$X_t = x + W_t + \int_0^t \frac{\nabla \rho}{2\rho}(X_s)\, ds + \int_0^\infty \int_0^t \nu_a(X_s)\, d\ell_s^a(\|X\|)\, \eta(da), \tag{1.2}$$

where $x \in \mathbb{R}^d$, W is a d-dimensional standard Brownian motion, ρ is typically a Muckenhoupt A_2-weight, ν_a is the unit outward normal on the boundary of the Euclidean ball of radius a about zero, $\ell^a(\|X\|)$ is the symmetric semimartingale local time at $a \in (0,\infty)$ of $\|X\|$, $\eta = \sum_{k\in\mathbb{Z}}(2\alpha_k - 1)\delta_{d_k}$ with $(\alpha_k)_{k\in\mathbb{Z}} \subset (0,1)$ is a sum of Dirac measures at a sequence $(d_k)_{k\in\mathbb{Z}} \subset (0,\infty)$ with exactly two accumulation points in $[0,\infty)$, one is zero and the other is $m_0 > 0$. More accumulation points could be allowed. For a discussion on this point we refer to [49, Remark 2.6(ii)]. The absolutely continuous component of drift $\frac{\nabla\rho}{2\rho}$ is typically unbounded and discontinuous. For an interpretation of the equation we refer to Theorem 2.1.6 and Remark 2.1.7. Variants of (1.2) with reflection on hyperplanes, instead of balls, but without accumulation points and Lipschitz drift appear in [67, 46, 47, 66]. In particular, (1.2) is a multidimensional analogue of an equation that was thoroughly studied in [49] and both equations share a lot of similarities. For instance the way to determine $(\gamma_k)_{k\in\mathbb{Z}}$ and $(\overline{\gamma}_k)_{k\in\mathbb{Z}}$ in (2.1) below, when $(\alpha_k)_{k\in\mathbb{Z}}$ in (1.2) is given, is the same here as in [49, Proposition 2.11]. The way to obtain $(\alpha_k)_{k\in\mathbb{Z}}$ from $(\gamma_k)_{k\in\mathbb{Z}}$ and $(\overline{\gamma}_k)_{k\in\mathbb{Z}}$ is described in Remark 2.1.7 (cf. also [49, Remark 2.4(ii) and Theorem 2.5]). See further [49, Remark 2.7] for another similarity and [49, Section 3.3] as well as references therein for a possible application to models with countably many permeable membranes that accumulate. For the construction of a solution to (1.2) for any starting point $x \in \mathbb{R}^d$ the key point is to identify (1.2) as distorted Brownian motion (see [3], [30] for an introduction to distorted Brownian motion). Then, one needs to show that the absolute continuity condition [35, p.165] is satisfied for the underlying Dirichlet form and that the strict Fukushima decomposition [35, Theorem 5.5.5] is applicable. In order to identify (1.2) as distorted Brownian motion we proceed informally as follows. We consider the density $\rho\phi$ for the underlying Dirichlet form in (2.2), where ϕ is a step function on annuli in $\mathbb{R}^d$, see (2.1), and ρ

is typically a Muckenhoupt A_2-weight (see Remark 2.1.1(iii) below). For the precise conditions we refer to (HY1)-(HY3) in Section 2.1. The logarithmic derivative, which is the drift of the distorted Brownian motion, is then informally given by

$$\frac{d(\rho\phi)}{\rho\phi} = \frac{d\rho}{\rho} + \frac{d\phi}{\phi}$$

(cf. [49, Remark 2.6]). This is rigorously performed through an integration by parts formula in Proposition 2.2.1 below. By results on Muckenhoupt weights in [62, 63], the existence of a jointly continuous transition kernel density for the semigroup associated to the Dirichlet form given in (2.2) is obtained. A Hunt process with the given transition kernel density is implicitly assumed to exist in condition (HY4) of Section 2.1 (for ways to construct such a process, we refer to Chapter 3, see also Remark 2.1.3). Under the conditions (HY1)-(HY4), we then show in a series of statements in Sections 2.3 and 2.4, that the strict Fukushima decomposition can be applied to obtain a solution to (1.2) (see main Theorem 2.1.6).
Finally one could think of generalizing (1.2), or more precisely the process of Theorem 2.1.6(i) with reflections on boundaries of Lipschitz domains (instead of smooth Euclidean balls). The main ingredient to obtain this generalization would be [68, Theorem 5.1] (see [70, Section 5]). In case of skew reflection at the boundary of a single $C^{1,\lambda}$-domain, $\lambda \in (0,1]$, $\rho \equiv 1$, and smooth diffusion coefficient, a weak solution has been constructed in [52, III. §3 and §4], see also references therein. However, the reflection term is defined as generalized drift and not explicitly as in Theorem 2.1.6.

Symmetric distorted Brownian motion

Let $E \subset \mathbb{R}^d$ and $\psi : E \to \mathbb{R}$ be a measurable function such that $\psi > 0$ dx-a.e. on E. In Chapter 3 we consider a regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E, \psi dx)$ that can be written as

$$\mathcal{E}(f,g) = \frac{1}{2}\int_E \nabla f \cdot \nabla g \, \psi \, dx, \quad f, g \in D(\mathcal{E}). \tag{1.3}$$

The regularity of $(\mathcal{E}, D(\mathcal{E}))$ provides existence of a Hunt process $\mathbb{M} = ((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E_\Delta})$ with lifetime ζ that is associated with $(\mathcal{E}, D(\mathcal{E}))$ and whose generator is informally given as

$$Lf = \frac{1}{2}\Delta f + \frac{\nabla\psi}{2\psi} \cdot \nabla f.$$

$\mathbb{M}$ is called distorted Brownian motion (cf. [3], [30], [31]) and forms as in (1.3) with infinitesimal generator L can be generalized to all kind of different state spaces E by finding an appropriate interpretation of the gradient ∇ and Laplacian Δ. Due to the good structural properties, like e.g. the self-adjointness of the corresponding generators, there is a huge literature about distorted Brownian motion in finite, as well as in infinite dimensions (see e.g. [9], [34], [53], [50], [5], [7] and references therein). We shall be concerned with a locally compact separable metric space E for our general results and with $E \subset \mathbb{R}^d$ like above in our concrete applications. The distorted Brownian motion has then typically an unbounded and discontinuous drift and of special interest is therefore the identification of the stochastic differential equation (hereafter SDE) that is fulfilled by it. It is well known how to identify the distorted Brownian motion for quasi-every starting point by using Fukushima's decomposition of additive functionals (see [30], [31], [35, Theorem 5.5.1], and [4], [44, Theorem 2.5] for infinite dimensional state space). This approach is in some sense abstract since the set of starting points that is excluded is not explicitly known and rather only given as a set of zero capacity. It can nonetheless be made explicit by looking at probability distributions $\mathbb{P}_\nu(\cdot) := \int_E \mathbb{P}_x(\cdot)\nu(dx)$ where ν is an explicitly given measure that does not charge sets of zero capacity. Another approach is to solve a corresponding martingale problem for as much as possible explicitly specified starting points (see [6], [11], [12], [27]). This may be a reasonable intermediate approach, especially if the functions for which the martingale problem is considered are dense in $D(\mathcal{E})$, but it does not lead directly to the identification of the SDE. Our strategy for the identification of the distorted Brownian motion for as much as possible explicitly specified starting points is based on Fukushima's absolute continuity condition and is known as the strict Fukushima decomposition (cf. [35, (4.2.9) and Theorem 5.5.5],[32], [33]). To our knowledge it is the first time it is applied systematically for weights $\psi \not\equiv const$. For some examples with $\psi \equiv const$, we refer to [13], [36] and [32], see also [35, Examples 5.2.2 and 5.5.3]. The strategy consists of two parts. The first one is to construct a Hunt process whose transition function has a density $p_t(x,y)$ w.r.t. the reference measure $m := \psi dx$ and is an m-version of the $L^2(E,m)$-semigroup $(T_t)_{t>0}$ associated with $(\mathcal{E}, D(\mathcal{E}))$, i.e. we need to construct a Hunt process $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E_\Delta})$ with life time ζ such that

$$P_t f(x) := \mathbb{E}_x[f(X_t)] = \int_E p_t(x,y)\, f(y)\, m(dy) \tag{1.4}$$

for any $t > 0, x \in E$, $f \in \mathcal{B}_b(E)$ and such that $P_t f$ is an m-version of $T_t f$ for any $f \in L^2(E,m) \cap \mathcal{B}_b(E)$ and $t > 0$. Note that even if $(T_t)_{t>0}$ is strong Feller, i.e. $T_t f$ has a continuous m-version for any $f \in \mathcal{B}_b(E)$ and $t > 0$, so that $T_t f$ has a density as in

(1.4), the process constructed via regularity by Dirichlet form methods does not satisfy this condition. In fact since such a Hunt process is only unique for quasi-every starting point (see [35, Theorem 4.2.8]), the absolute continuity condition may be violated for some points $x \in E$ in a capacity zero set. For the construction of a Hunt process $\mathbb{M}$ on a general locally compact separable metric space E that satisfies the absolute continuity condition, we use two methods. The first one is the well known Feller semigroup method that we summarize in Section 3.1.1 and that we apply in the form of Lemma 3.1.3. It turns out that one can use heat kernel estimates to verify the conditions of Lemma 3.1.3 for concrete Muckenhoupt weights (cf. Remark 3.1.4). The second method which is developed in Section 3.1.1 is what we call the Dirichlet form method and it is a refinement of the method introduced in [6, Section 4]. Our contribution here is to exploit the structure of a carré du champ (see Lemma 3.1.5 and Remark 3.1.7(ii)). For other work, where the Dirichlet form method of [6] is adopted, we refer to [12, 11]. As in the case of Feller semigroups, we apply these general results in Section 3.2 to concrete Muckenhoupt A_2 weights (see Lemma 3.2.6(i) and Propositions 3.2.13, 3.2.17). We remark that it remains open whether the absolute continuity condition holds for general Muckenhoupt A_2 weights or not. According to Proposition 3.2.3(i) and (iii), when using the Feller method it remains to show Lemma 3.1.3(i), and according to Proposition 3.2.3(i) and (ii), when using the Dirichlet form method it remains to show (**H2**)$'$(i) and (ii). In Section 3.3, we obtain the absolute continuity condition from results of [6] using the appropriate part Dirichlet form (see Lemma 3.3.2). In Section 3.4, we assume the absolute continuity condition to be verified, but refer to [11] to which it accordingly holds under certain conditions (see Remark 3.4.2). The results of Section 3.4 are also achieved by specifying the appropriate part Dirichlet form (see Lemma 3.4.3). The necessary tools for part Dirichlet forms and general auxiliary results are presented in Section 3.1.2.

The second part of the strategy consists in finding good estimates for the drift potentials

$$R_1\mu(x) = \int_E r_1(x,y)\,\mu(dy)$$

corresponding to the logarithmic derivative $\mu := \frac{\nabla\psi}{2\psi}$ in the sense of distributions and to measures μ on ∂E that occur through integration by parts as boundary terms in case of existing boundary ∂E. Here $r_1(x,y) = \int_0^\infty e^{-t} p_t(x,y)\,dt$. Concretely, in Section 3.2, we consider Muckenhoupt A_2 weights $\psi = \rho\phi$, where ρ is a weakly differentiable function and ϕ is a function that is piecewise constant and has discontinuities along boundaries of Euclidean balls (see (3.28)), along the boundary of a Lipschitz domain (see (3.30)) and along hyperplanes (see (3.32)). In this case using informally Leibniz rule for $\nabla(\rho\phi)$, we see that $\mu(dy)$ is given as the sum of the absolutely continuous part $\frac{\nabla\rho}{2\rho}(y)\, m(dy)$ and the corresponding boundary measures. In Section 3.3, we consider the case where $\phi \equiv 1$ and E has no boundary so that $\mu(dy) = \frac{\nabla\rho}{2\rho}(y)\, m(dy)$ and in Section 3.4, we consider the case where $\phi \equiv 1$ and existing boundary, so that μ is given as the sum of $\frac{\nabla\rho}{2\rho}(y)\, m(dy)$ and a weighted surface measure (see Lemma 3.4.7). Our key for estimating potentials is Proposition 3.1.14 that we found very useful and apply throughout the chapter. Especially, if no continuity properties of a potential are known, we use resolvent kernel estimates to find continuous Riesz potentials (see (3.22) and Lemma 3.2.5) as upper bound r_1^G as in Proposition 3.1.14 for the potential, i.e. we use Proposition 3.1.14 in combination with resolvent kernel estimates and Lemma 3.2.5. We use this procedure for instance globally in Lemma 3.2.6(iii)-(v) where for the global resolvent kernel estimates, we use known global heat kernel estimates for Muckenhoupt weights from [63] (see (3.15)). We use it locally in Lemma 3.4.8 using local heat kernel estimates that we derive using Nash type inequalities and the Davies method of [19] similarly to what is done in [13, Theorems 2.3, 3.1] (see Lemma 3.4.4, Proposition 3.4.5 and Corollary 3.4.6). Of special interest could be the corresponding localization procedure via part processes that we apply on a nice exhaustive sequence of sets for the state space (see conditions (ι), (κ) in Section 3.4, Lemma 3.4.8, Proposition 3.4.9, Lemmas 3.4.10, 3.4.11 and proof of Theorem 3.4.12). We use it when global resolvent kernel estimates do not provide enough regularity or are not at hand. For other places in this chapter where we use this localization procedure see Proposition 3.2.8(ii), Theorem 3.2.9(ii) and Remark 3.2.16.

The Muckenhoupt A_2 weights $\psi = \rho\phi$ that we investigate in Section 3.2, lead to solutions of SDEs of the following type

$$X_t = x + W_t + \int_0^t \frac{\nabla\rho}{2\,\rho}(X_s)\, ds + L_t, \quad t \geq 0,\ x \in E \subset \mathbb{R}^d, \tag{1.5}$$

where L may be a series of local times (see (3.21) of Theorem 3.2.4). Theorem 3.2.4 is formulated under general conditions on ρ and ϕ. We then extensively study the typical case of an A_2 weight where $\rho(x) = \|x\|^\alpha$, $\alpha \in (-d+1, d)$ and ϕ is an explicitly given piecewise constant function that is globally bounded above and below by strictly positive constants. In this case it is known that the capacity of $\{0\}$ is zero, iff $\alpha \in [-d+2, d)$. We obtain that one can choose $E = \mathbb{R}^d$, if $\alpha \in (-d+1, 2)$ and $L \equiv 0$, or if $\alpha \in (-d+1, 1)$ and $L \not\equiv 0$ (see Proposition 3.2.8(i), Theorem 3.2.9(i) and Theorem 3.2.15) and that one can choose $E = \mathbb{R}^d \setminus \{0\}$ in the remaining cases

(see Proposition 3.2.8(ii), Theorem 3.2.9(ii) and Remark 3.2.16). Two observations are here worth to be noted. The first is that we are able to start in 0 although $\{0\}$ might be a capacity zero set and the second is that we lose one dimension in case there are boundary terms. The reason for the last is that we use continuous Riesz potentials of the form (3.22) as upper bounds for our drift potentials and that drifts which are given as surface measures on a nice boundary are equivalent to the Lebesgue measure of one dimension less (cf. Lemma 3.2.6(v)). The concrete examples of drifts L that we obtain in (1.5) can be summarized as follows. If ϕ is as in (3.28) piecewise constant on countably many annuli with jumps along their boundaries, L is given as the last term in (3.29) which corresponds to a distorted Brownian motion with skew reflection on the boundary of Euclidean balls that may accumulate. (3.29) seems new to us. We could not find any similar equation in multi-dimensions in the literature. Its one-dimensional counterpart is studied extensively in [49]. If ϕ is as in (3.30) piecewise constant on a bounded Lipschitz domain and on its complement, then L is given as a scalar multiple of the boundary local time on the boundary of a Lipschitz domain G as in (3.31). The corresponding process could be called a β-skew distorted Brownian motion w.r.t. G. In case of skew reflection at the boundary of a $C^{1,\lambda}$-domain, $\lambda \in (0,1]$ and smooth diffusion coefficient, a weak solution has been constructed in [52, III. §3 and §4], see also references therein. The reflection term in [52] is defined as generalized drift and not explicitly as in (3.29). If ϕ is as in (3.32) piecewise constant on countably many infinite strips with jumps along countably many hyperplanes, then L is given as the last term in (3.36). Variants of (3.36), but without accumulation points and Lipschitz drift appear in [67, 47, 66]. For recent related work, we refer to [2].
In Section 3.3, we complete results of [6]. There the distorted Brownian motion was constructed on $\mathbb{R}^d \setminus \{\psi = 0\}$ for certain weights ψ (cf. Section 3.3), but the corresponding SDE was not identified. It was noted in [6, Remark 5.6] that refining arguments from [35] one could possibly achieve this identification. As already mentioned, we do this using the part Dirichlet form on $\mathbb{R}^d \setminus \{\psi = 0\}$. For details we refer to Section 3.3.
In Section 3.4, we complete results from [68]. Precisely, under the assumptions $(\eta)-(\kappa)$ of Section 3.4, we show in Theorem 3.4.12 that the Skorokhod decomposition that was obtained in [68] for quasi-every starting point can be achieved in concrete examples for every starting point outside an explicitly specified capacity zero set in the symmetric case. We note that the absolute continuity condition is assumed to hold in (θ). For additional conditions according to which the absolute continuity condition is satisfied, we refer to [11] (see Remark 3.4.2). For work that is strongly related with Theorem 3.4.12, we refer to [13, 21, 36, 51].

Finally, let us remark that we only treat the semimartingale case, but that the strict Fukushima decomposition has also been formulated in the non-semimartingale case (see [32]). It could be interesting to see which phenomena occur in this case. Moreover, because we did not want to overload this chapter, we consider the (a_{ij})-case in Chapter 5. The drift potentials that occur in the (a_{ij})-case can be handled by almost same methods that are presented here.

Non-symmetric distorted Brownian motion

In Chapter 4 we are concerned with the non-symmetric Dirichlet form given by (the closure of)

$$\mathcal{E}(f,g) := \frac{1}{2}\int_{\mathbb{R}^d} \langle \nabla f, \nabla g\rangle \, \mathrm{d}m - \int_{\mathbb{R}^d} \langle B, \nabla f\rangle g \, \mathrm{d}m \ , \quad f,g \in C_0^\infty(\mathbb{R}^d) \ , \tag{1.6}$$

on $L^2(\mathbb{R}^d, m)$, $m := \rho \, \mathrm{d}x$, and the corresponding stochastic differential equation (SDE)

$$X_t = x + W_t + \int_0^t \left(\frac{\nabla \rho}{2\rho} + B\right)(X_s) \, \mathrm{d}s \ , \quad t < \zeta \ , \tag{1.7}$$

where $x \in \{\tilde{\rho} > 0\}$, $\tilde{\rho}$ is the continuous version of ρ (which exists as a consequence of (HS1)), and ζ is the lifetime (=explosion time). Our conditions on ρ and B are formulated as Hypotheses (HS1)-(HS3) in Section 4.1 below.

It is well known that starting with (1.6) by Dirichlet form theory one can construct a weak solution to (1.7) for quasi-every starting point $x \in \mathbb{R}^d$, and usually there is no analytic characterization (in terms of ρ and B) of the set of “allowed” starting points. In case $B \equiv 0$, it was however shown in [6] (see also [10],[27], for extensions of this result to other situations), that (1.7) has a weak solution for every $x \in \{\tilde{\rho} > 0\}$ in the sense of the martingale problem, and that for such starting points the process X_t stays in $\{\tilde{\rho} > 0\}$ before its lifetime ζ. The identification of (1.7) with $B \equiv 0$ for any $x \in \{\tilde{\rho} > 0\}$ in the sense of a weak solution of an SDE related to the form in (1.6) has been worked out as a part of a general framework in Section 3.3.

The first aim of Chapter 4 is to generalize these results to $B \not\equiv 0$, i.e. to the non-symmetric case (see Remark 4.1.2). The proof follows ideas from [6], but requires some modifications. For example, one observation is that the elliptic regularity results in weighted spaces from [6] extend to the non-symmetric case. The corresponding result is formulated as Theorem 4.2.7 in Section 4.2 below.

It is well known by [43, Theorem 2.1] (see also [29], [73]) that for every $x \in \{\tilde{\rho} > 0\}$ there exists a strong solution (i.e. adapted to the filtration generated by $(W_t)_{t\geq 0}$) to (1.7), which is pathwise and weak unique. Hence this solution coincides with our weak solution (which is hence a strong solution) from Theorem 4.2.7. Thus we have identified the Dirichlet form associated to the Markov processes, given by the laws $\mathbb{P}_x, x \in \{\tilde{\rho} > 0\}$, of these strong solutions, to be the closure of (1.6). As a consequence, we can apply the theory of Dirichlet forms to obtain further properties of the solutions to (1.7) for every starting point in $\{\tilde{\rho} > 0\}$.

In Chapter 4, as our second aim, we concentrate on proving non-explosion results for (1.7) using Dirichlet form theory, which means (cf. Remark 4.1.13) that the process started in $x \in \{\tilde{\rho} > 0\}$ will neither go to infinity nor hit any point in $\{\tilde{\rho} = 0\}$ in finite time. Non-explosion criteria from Dirichlet form theory are of analytic nature and different from the usual ones known from the theory of SDE (e.g. the one proved in [43], see Remark 4.3.2 (ii) below), but very useful in applications.

Finally, we present a number of concrete applications where the density $\rho \left(= \frac{\mathrm{d}m}{\mathrm{d}x}\right)$ is in certain Muckenhoupt classes. Our main result here is Theorem 4.4.5.

The organization of Chapter 4 is as follows. After this introduction in Section 4.1 we recall some important elliptic regularity results for the Kolmogorov operator corresponding to (1.7), i.e. the generator of the Dirichlet form (1.6), under the assumption (HS1) on ρ and (HS2) on B. Subsequently, we present their analytic consequences associated to the closure of (1.6). In Section 4.2 we construct the weak solutions of (1.7) for every $x \in \{\tilde{\rho} > 0\}$. In Section 4.3 we show that by [43, Theorem 2.1] these solutions are strong, pathwise and weak unique. Section 4.4 is devoted to the mentioned applications.

Degenerate elliptic forms w.r.t. 2-admissible weights

In Chapter 5 we consider the regular symmetric Dirichlet form given by (the closure of) symmetric bilinear form

$$\mathcal{E}(f,g) = \frac{1}{2} \int_{\mathbb{R}^d} \langle A \, \nabla f, \nabla g \rangle \, dx, \quad f, g \in C_0^\infty(\mathbb{R}^d), \tag{1.8}$$

where $A = (a_{ij})_{1 \leq i,j \leq d}$ is a symmetric degenerate elliptic $d \times d$ matrix, that is $a_{ij} \in L^1_{loc}(\mathbb{R}^d, dx)$ and there exists a constant $\lambda \in [1, 2)$ such that for dx-a.e. $x \in \mathbb{R}^d$

$$\lambda^{-1} \, \rho(x) \, \|\xi\|^2 \leq \langle A(x)\xi, \xi \rangle \leq \lambda \, \rho(x) \, \|\xi\|^2, \quad \forall \xi \in \mathbb{R}^d.$$

We construct a solution to the stochastic differential equation corresponding to the Dirichlet form (1.8), i.e. for any $x \in \mathbb{R}^d$, $i = 1, \dots, d$ $\mathbb{P}_x$-a.s.

$$X_t^i = x^i + \sum_{j=1}^d \int_0^t \frac{\sigma_{ij}}{\sqrt{\rho}}(X_s)\, dW_s^j + \frac{1}{2}\sum_{j=1}^d \int_0^t \frac{\partial_j a_{ij}}{\rho}(X_s)\, ds, \quad t \geq 0,$$

where $(\sigma_{ij})_{1 \leq i,j \leq d} = \sqrt{A}$ is the positive square root of the matrix A, $W = (W^1, \dots, W^d)$ is a standard d-dimensional Brownian motion on $\mathbb{R}^d$.

1.2 Notation

For a locally compact separable metric space $(E, \mathbf{d})$ with Borel σ-algebra $\mathcal{B}(E)$ we denote the set of all $\mathcal{B}(E)$-measurable $f : E \to \mathbb{R}$ which are bounded, or nonnegative by $\mathcal{B}_b(E)$, $\mathcal{B}^+(E)$ respectively. Let $B_r(y) := \{x \in E \mid \mathbf{d}(x,y) < r\}$, $r > 0$, $y \in E$. The usual L^q-spaces $L^q(E,\mu)$, $q \in [1,\infty]$ are equipped with L^q-norm $\|\cdot\|_q$ with respect to the measure μ on E, $\mathcal{A}_b : = \mathcal{A} \cap \mathcal{B}_b(E)$ for $\mathcal{A} \subset L^q(E,\mu)$, and $L^q_{loc}(E,\mu) := \{f \mid f \cdot 1_U \in L^q(E,\mu), \forall U \subset E, U \text{ relatively compact open}\}$, where 1_A denotes the indicator function of a set A. If $\mathcal{A}$ is a set of functions $f : E \to \mathbb{R}$, we define $\mathcal{A}_0 := \{f \in \mathcal{A} \mid \mathrm{supp}(f) : = \mathrm{supp}(|f|dm) \text{ is compact in } E\}$. As usual, we also denote the set of continuous functions on E, the set of continuous bounded functions on E, the set of compactly supported continuous functions in E by $C(E)$, $C_b(E)$, $C_0(E)$, respectively. $C_\infty(E)$ denotes the space of continuous functions on E which vanish at infinity. For $A \subset E$ let $\overline{A}$ denote the closure of A in E.

Let $\nabla f := (\partial_1 f, \dots, \partial_d f)$ and $\Delta f := \sum_{j=1}^d \partial_{jj} f$ where $\partial_j f$ is the j-th weak partial derivative of f and $\partial_{jj} f := \partial_j(\partial_j f)$, $j = 1, \dots, d$. For any open set $E \subset \mathbb{R}^d$ we denote $C^\infty(E)$, $C_0^\infty(E)$ by the set of all infinitely differentiable functions on E, the set of all infinitely differentiable functions with compact support in E, respectively. Furthermore for any relatively compact open set $E \subset \mathbb{R}^d$ $C(\overline{E}) := \{ f : \overline{E} \to \mathbb{R} \mid \exists g \in C(\mathbb{R}^d) \textit{ with } f = g \textit{ on } \overline{E}\}$ and $C^\infty(\overline{E}) := \{ f : \overline{E} \to \mathbb{R} \mid \exists g \in C^\infty(\mathbb{R}^d) \textit{ with } f = g \textit{ on } \overline{E}\}$. As usual dx denotes Lebesgue measure on $\mathbb{R}^d$ and for any open set $E \subset \mathbb{R}^d$ the Sobolev space $H^{1,q}(E,dx)$, $q \geq 1$ is defined to be the set of all functions $f \in L^q(E,dx)$ such that $\partial_j f \in L^q(E,dx)$, $j = 1, \dots, d$, and $H^{1,q}_{loc}(E,dx) := \{f \mid f \cdot \varphi \in H^{1,q}(E,dx), \forall \varphi \in C_0^\infty(E)\}$. We always equip $\mathbb{R}^d$ with the Euclidean norm $\|\cdot\|$ with corresponding inner product $\langle \cdot, \cdot \rangle$ and write $B_r(x) := \{y \in \mathbb{R}^d \mid \|x - y\| < r\}$, $x \in \mathbb{R}^d$, $r > 0$.

Chapter 2

Distorted skew Brownian motion w.r.t. discontinuous Muckenhoupt weights

2.1 Formulation of the main theorem

Let $d \geq 2$, $m_0 \in (0, \infty)$ and $(l_k)_{k \in \mathbb{Z}} \subset (0, m_0)$, $0 < l_k < l_{k+1} < m_0$, be a sequence converging to 0 as $k \to -\infty$ and converging to m_0 as $k \to \infty$. Let $(r_k)_{k \in \mathbb{Z}} \subset (m_0, \infty)$, $m_0 < r_k < r_{k+1} < \infty$, be a sequence converging to m_0 as $k \to -\infty$ and tending to infinity as $k \to \infty$. Let

$$\phi := \sum_{k \in \mathbb{Z}} \left(\gamma_k \cdot 1_{A_k} + \overline{\gamma}_k \cdot 1_{\hat{A}_k} \right), \tag{2.1}$$

where γ_k , $\overline{\gamma}_k \in (0, \infty)$, $A_k := B_{l_k} \setminus \overline{B}_{l_{k-1}}$, $\hat{A}_k := B_{r_k} \setminus \overline{B}_{r_{k-1}}$, $B_r := \{x \in \mathbb{R}^d \mid \|x\| < r\}$ for any $r > 0$. We denote by $d\sigma_r$ the surface measure on the boundary ∂B_r of B_r, $r > 0$. A function $\psi \in \mathcal{B}(\mathbb{R}^d)$ with $\psi > 0$ dx-a.e. is said to be a Muckenhoupt A_2-weight (in notation $\psi \in A_2$), if there exists a positive constant A such that, for every ball $B \subset \mathbb{R}^d$,

$$\left(\int_B \psi dx \right) \left(\int_B \psi^{-1} dx \right) \leq A \left(\int_B 1 \, dx \right)^2 .$$

For more on Muckenhoupt weights, we refer to [72].
Throughout this chapter we shall assume

(HY1) $\sum_{k\in\mathbb{Z}} |\,\gamma_{k+1} - \gamma_k\,| + \sum_{k\le 0} |\,\overline{\gamma}_{k+1} - \overline{\gamma}_k\,| < \infty$ and for all $r > 0$ there exists $\delta_r > 0$ such that $\phi \ge \delta_r$ dx-a.e. on B_r.

(HY2) $\rho\,\phi \in A_2$.

Remark 2.1.1. *(i) (HY1) implies that $\phi \in L^1_{loc}(\mathbb{R}^d, dx)$ and that $\gamma := \lim_{k\to\infty} \gamma_k$, $\overline{\gamma} := \lim_{k\to-\infty} \overline{\gamma}_k$ exist and $\gamma > 0$, $\overline{\gamma} > 0$. In particular, ϕ is locally bounded above and locally bounded away from zero.*
(ii) (HY1) and (HY2) imply $\rho > 0$ dx-a.e.
(iii) Let $c > 0$. If $c^{-1} \le \phi \le c$ and $\rho \in A_2$, or if $\rho = 1$ and $c^{-1}\,\psi \le \phi \le c\,\psi$ for some $\psi \in A_2$, then $\rho\,\phi \in A_2$.

Furthermore, we shall throughout this chapter assume the following condition.

(HY3) $\rho = \xi^2$ for some $\xi \in H^{1,2}_{loc}(\mathbb{R}^d, dx)$.

Remark 2.1.2. *(HY3) implies that $\rho \in H^{1,1}_{loc}(\mathbb{R}^d, dx)$ and by (HY1) $\frac{\|\nabla\rho\|}{\rho} \in L^2_{loc}(\mathbb{R}^d,\ \rho\phi\ dx)$ since ϕ is locally bounded above dx-a.e.*

We consider the symmetric positive definite bilinear form

$$\mathcal{E}(f, g) := \frac{1}{2}\int_{\mathbb{R}^d} \nabla f \cdot \nabla g\ (\rho\,\phi)\ dx, \quad f, g \in C_0^\infty(\mathbb{R}^d). \tag{2.2}$$

Since $\rho\,\phi \in A_2$, we have $\frac{1}{\rho\,\phi} \in L^1_{loc}(\mathbb{R}^d, dx)$, and the latter implies that (2.2) is closable in $L^2(\mathbb{R}^d, \rho\phi\,dx)$ (see [44, II.2 a)]). The closure $(\mathcal{E}, D(\mathcal{E}))$ of (2.2) is a strongly local, regular, symmetric Dirichlet form (cf. e.g. [63, p. 274]). As usual we define $\mathcal{E}_1(f, g) := \mathcal{E}(f, g) + (f, g)_{L^2(\mathbb{R}^d,\,\rho\phi\,dx)}$ for $f, g \in D(\mathcal{E})$ and $\|\,f\,\|_{D(\mathcal{E})} := \mathcal{E}_1(f, f)^{1/2}$, $f \in D(\mathcal{E})$. Let $(T_t)_{t\ge0}$ and $(G_\alpha)_{\alpha>0}$ be the $L^2(\mathbb{R}^d, \rho\phi\,dx)$-semigroup and resolvent associated to $(\mathcal{E}, D(\mathcal{E}))$ and $(L, D(L))$ be the corresponding generator (see [44, Diagram 3, p. 39]). From [62, p. 303 Proposition 2.3] and [63, p. 286 A)] we know that there exists a jointly continuous transition kernel density $p_t(x, y)$ such that

$$P_t f(x) := \int_{\mathbb{R}^d} p_t(x, y)\, f(y)\ \rho(y)\ \phi(y)\, dy\,, \quad t > 0, \quad x \in \mathbb{R}^d,$$

$f \in \mathcal{B}_b(\mathbb{R}^d)$, is a $\rho\phi dy$-version of $T_t f$ if $f \in L^2(\mathbb{R}^d, \rho\,\phi\,dx) \cap \mathcal{B}_b(\mathbb{R}^d)$. Furthermore, taking the Laplace transform of $p_t(x, y)$ there exists a resolvent kernel density $r_\alpha(x, y)$ such that

$$R_\alpha f(x) := \int_{\mathbb{R}^d} r_\alpha(x, y)\, f(y)\ \rho(y)\ \phi(y)\, dy\,, \quad \alpha > 0, \quad x \in \mathbb{R}^d,$$

$f \in \mathcal{B}_b(\mathbb{R}^d)$, is a $\rho\phi dy$-version of $G_\alpha f$ if $f \in L^2(\mathbb{R}^d, \rho\,\phi\,dx) \cap \mathcal{B}_b(\mathbb{R}^d)$. Accordingly, for a signed Radon measure μ, let us define

$$R_\alpha\mu(x) = \int_{\mathbb{R}^d} r_\alpha(x,y)\,\mu(dy)\,, \quad \alpha > 0, \quad x \in \mathbb{R}^d,$$

whenever this makes sense. Since $p_t(\cdot,\cdot)$ is jointly continuous and $p_t(x,y)$ has exponential decay in y for fixed t and x in a compact set, $(P_t)_{t\geq 0}$ is strong Feller, i.e. $P_t(\mathcal{B}_b(\mathbb{R}^d)) \subset C_b(\mathbb{R}^d)$. For details, we refer to Chapter 3. In particular $R_1(\mathcal{B}_b(\mathbb{R}^d)) \subset C_b(\mathbb{R}^d)$.
We consider further the following condition

(HY4) There exists a Hunt process

$$\mathbb{M} = (\mathbf{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \zeta, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d\cup\{\Delta\}}),$$

with state space $\mathbb{R}^d$ and lifetime ζ, which has $(P_t)_{t\geq 0}$ as transition function, $R_1 f$ is continuous for any $f \in L^2(\mathbb{R}^d, \rho\phi dx)$ with compact support, and if $\phi \not\equiv$ const., we additionally assume $R_1(1_G\ \rho d\sigma_r)$ is continuous for any $G \subset \mathbb{R}^d$ relatively compact open, $r > 0$.

In (HY4), Δ is the cemetery point and as usual any function $f : \mathbb{R}^d \to \mathbb{R}$ is extended to $\{\Delta\}$ by setting $f(\Delta) := 0$.

Remark 2.1.3. *A Hunt process associated with $(P_t)_{t\geq 0}$ as in (HY4) can be constructed by the same methods as used in [6, Section 4]. The method used in [6, Section 4] is applicable, if one can find enough nice functions in $D(L)$ (cf. [6, proof of Lemma 4.6]). A Hunt process with transition function $(P_t)_{t\geq 0}$ as in (HY4) can also be constructed by showing that $(P_t)_{t\geq 0}$ defines a classical Feller semigroup. For details and concrete examples we refer to Chapter 3.*

Under (HY4), $\mathbb{M}$ satisfies in particular the *absolute continuity condition* as stated in [35, p. 165].

Proposition 2.1.4. *Let a Dirichlet form be given as the closure of*

$$\frac{1}{2}\int_{\mathbb{R}^d} \nabla f \cdot \nabla g\ \psi\,dx, \quad f,g \in C_0^\infty(\mathbb{R}^d)$$

in $L^2(\mathbb{R}^d, \psi\,dx)$ where $\psi \in A_2$. Then it is conservative.

Proof. By [72, Proposition 1.2.7] $\psi\, dx$ is volume doubling. Hence by [38, Proposition 5.1, Proposition 5.2]

$$c_1\, r^{\alpha'} \le \psi\, dx(B_r) \le c_2\, r^{\alpha}, \quad \forall r \ge 1,$$

where $c_1, c_2, \alpha, \alpha' > 0$ are constants. In particular,

$$\int_1^\infty \frac{r}{\log\big(\psi\, dx(B_r)\big)}\, dr = \infty.$$

Hence conservativeness follows by [61, Theorem 4]. $\square$

Remark 2.1.5. *Proposition 2.1.4 holds more generally for A_p-weights, $p \in [1,\infty)$ (see [72, Definitoin 1.2.2] for the definition of A_p) by the same arguments as in the proof of Proposition 2.1.4.*

It follows from Proposition 2.1.4 and the strong Feller property of $(P_t)_{t\ge 0}$ that under (HY4)

$$\mathbb{P}_x(\zeta = \infty) = 1, \quad \forall x \in \mathbb{R}^d. \tag{2.3}$$

We will refer to [35] from now on till the end of this chapter, hence some of its standard notations may be adopted below without definition. The following theorem is the main result of this chapter. It will be proved in Section 2.4.

Theorem 2.1.6. *Suppose (HY1)-(HY4) hold. Then:*
(i) The process $\mathbb{M}$ satisfies

$$X_t = x + W_t + \int_0^t \frac{\nabla \rho}{2\rho}(X_s)\, ds + N_t\ , \quad t \ge 0, \tag{2.4}$$

$\mathbb{P}_x$ *-a.s. for any $x \in \mathbb{R}^d$, where W is a standard d-dimensional Brownian motion starting from zero and*

$$N_t := \sum_{k\in\mathbb{Z}} \left(\frac{\gamma_{k+1}-\gamma_k}{\gamma_{k+1}+\gamma_k} \int_0^t \nu_{l_k}(X_s)\, d\ell_s^{l_k} + \frac{\overline{\gamma}_{k+1}-\overline{\gamma}_k}{\overline{\gamma}_{k+1}+\overline{\gamma}_k} \int_0^t \nu_{r_k}(X_s)\, d\ell_s^{r_k} \right)$$

$$+ \frac{\overline{\gamma}-\gamma}{\overline{\gamma}+\gamma} \int_0^t \nu_{m_0}(X_s)\, d\ell_s^{m_0},$$

where $\nu_r = (\nu_r^1, \dots, \nu_r^d)$, $r > 0$ is the unit outward normal vector on ∂B_r and ℓ^{l_k}, ℓ^{r_k} and ℓ^{m_0} are boundary local times of X, i.e. they are positive continuous additive functionals *of X in the strict sense associated via the Revuz correspondence (cf. [35, Theorem 5.1.3]) with the weighted surface measures $\frac{\gamma_{k+1}+\gamma_k}{2}\,\rho\, d\sigma_{l_k}$ on ∂B_{l_k}, $\frac{\overline{\gamma}_{k+1}+\overline{\gamma}_k}{2}\,\rho\, d\sigma_{r_k}$ on ∂B_{r_k}, and $\frac{\overline{\gamma}+\gamma}{2}\,\rho\, d\sigma_{m_0}$ on ∂B_{m_0}, respectively, and related via the formulas*

$$\begin{aligned}
\mathbb{E}_x\left[\int_0^\infty e^{-t}\, d\ell_t^{l_k}\right] &= R_1\left(\frac{\gamma_{k+1}+\gamma_k}{2}\,\rho d\sigma_{l_k}\right)(x),\\
\mathbb{E}_x\left[\int_0^\infty e^{-t}\, d\ell_t^{r_k}\right] &= R_1\left(\frac{\overline{\gamma}_{k+1}+\overline{\gamma}_k}{2}\,\rho d\sigma_{r_k}\right)(x),\\
\mathbb{E}_x\left[\int_0^\infty e^{-t}\, d\ell_t^{m_0}\right] &= R_1\left(\frac{\overline{\gamma}+\gamma}{2}\,\rho d\sigma_{m_0}\right)(x),
\end{aligned}$$

which all hold for any $x \in \mathbb{R}^d$, $k \in \mathbb{Z}$.
(ii) $\big((\|X_t\|)_{t\geq 0}, \mathbb{P}_x\big)$ is a continuous semimartingale for any $x \in \mathbb{R}^d$ and

$$\mathbb{P}_x\big(\ell_t^a = \ell_t^a(\|X\|)\big) = 1, \quad \forall x \in \mathbb{R}^d,\ t \geq 0,\ a \in \{m_0, l_k, r_k : k \in \mathbb{Z}\},$$

where $\ell_t^a(\|X\|)$ is the symmetric semimartingale local time of $\|X\|$ at $a \in (0,\infty)$ as defined in [54, VI.(1.25)].

Remark 2.1.7. *In view of Theorem 2.1.6, the non absolutely continuous drift component N in (2.4) may be interpreted as follows. Define*

$$\alpha_k := \frac{\gamma_{k+1}}{\gamma_{k+1}+\gamma_k}, \quad k \in \mathbb{Z}.$$

Then $\alpha_k \in (0,1)$ and

$$\frac{\gamma_{k+1}-\gamma_k}{\gamma_{k+1}+\gamma_k} = 2\alpha_k - 1 = \alpha_k - (1-\alpha_k).$$

and so the drift component

$$\frac{\gamma_{k+1}-\gamma_k}{\gamma_{k+1}+\gamma_k}\int_0^t \nu_{l_k}(X_s)\, d\ell_s^{l_k}(\|X\|)$$

corresponds to an outward normal reflection with probability α_k and an inward normal reflection with probability $1-\alpha_k$ when X_t hits ∂B_{l_k} (cf. [70]). Analogous interpretations hold for the other reflection terms. Thus ∂B_{l_k}, ∂B_{r_k} and ∂B_{m_0} can be seen as boundaries where a skew reflection takes place, or alternatively as permeable membranes.

2.2 Integration by parts formula

Proposition 2.2.1. *Suppose (HY1)-(HY3) hold. The following integration by parts formula holds for* $f, g \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned} -\mathcal{E}(f,g) &= \int_{\mathbb{R}^d} \left(\frac{1}{2}\Delta f + \nabla f \cdot \frac{\nabla\rho}{2\rho}\right) g\,\rho\,\phi\,dx + \frac{\overline{\gamma}-\gamma}{2}\int_{\partial B_{m_0}} \nabla f \cdot \nu_{m_0}\, g\,\rho\, d\sigma_{m_0} \\ &+ \sum_{k\in\mathbb{Z}} \left(\frac{\gamma_{k+1}-\gamma_k}{2} \int_{\partial B_{l_k}} \nabla f \cdot \nu_{l_k}\, g\,\rho\, d\sigma_{l_k} + \frac{\overline{\gamma}_{k+1}-\overline{\gamma}_k}{2} \int_{\partial B_{r_k}} \nabla f \cdot \nu_{r_k}\, g\,\rho\, d\sigma_{r_k} \right). \end{aligned}$$

The last summation is in particular only over finitely many r_k*,* $k \geq 1$*, since* f *has compact support.*

Proof. For $f, g \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned} &\mathcal{E}(f,g) = \frac{1}{2}\sum_{j=1}^{d} \int_{\mathbb{R}^d} \partial_j f \partial_j g \ (\rho\,\phi)\ dx \\ = \ & -\frac{1}{2}\sum_{j=1}^{d}\sum_{k\in\mathbb{Z}} \left(\gamma_k \int_{A_k} \left(\partial_{jj} f + \partial_j f\,\frac{\partial_j \rho}{\rho}\right) g\,\rho\,dx \,+\, \overline{\gamma}_k \int_{\hat{A}_k} \left(\partial_{jj} f + \partial_j f\,\frac{\partial_j \rho}{\rho}\right) g\,\rho\,dx \right) \\ &+\frac{1}{2}\sum_{j=1}^{d}\sum_{k\in\mathbb{Z}} \left(\int_{A_k} \gamma_k\, \partial_j\left(\partial_j f\ g\ \rho\right)\ dx + \int_{\hat{A}_k} \overline{\gamma}_k\, \partial_j\left(\partial_j f\ g\ \rho\right)\ dx \right). \end{aligned}$$

The first term equals

$$-\frac{1}{2}\int_{\mathbb{R}^d} \left(\Delta f + \nabla f \cdot \frac{\nabla\rho}{\rho}\right) g\,\rho\,\phi\,dx,$$

and the second term equals

$$\begin{aligned} &\frac{1}{2}\sum_{j=1}^{d}\sum_{k\in\mathbb{Z}} \left(\int_{\partial B_{l_k}} \gamma_k\, \partial_j f\, \nu^j_{l_k}\, g\,\rho\, d\sigma_{l_k} - \int_{\partial B_{l_{k-1}}} \gamma_k\, \partial_j f\, \nu^j_{l_{k-1}}\, g\,\rho\, d\sigma_{l_{k-1}} \right) \\ &+\frac{1}{2}\sum_{j=1}^{d}\sum_{k\in\mathbb{Z}} \left(\int_{\partial B_{r_k}} \overline{\gamma}_k\, \partial_j f\, \nu^j_{r_k}\, g\,\rho\, d\sigma_{r_k} - \int_{\partial B_{r_{k-1}}} \overline{\gamma}_k\, \partial_j f\, \nu^j_{r_{k-1}}\, g\,\rho\, d\sigma_{r_{k-1}} \right) \end{aligned}$$

$$
\begin{aligned}
= \quad & -\frac{1}{2}\left(\lim_{k\to-\infty}\int_{\partial B_{l_{k-1}}} \gamma_k\,\nabla f\cdot\nu_{l_{k-1}}\,g\,\rho\,d\sigma_{l_{k-1}} + \sum_{k\in\mathbb{Z}}\int_{\partial B_{l_k}} (\gamma_{k+1}-\gamma_k)\,\nabla f\cdot\nu_{l_k}\,g\,\rho\,d\sigma_{l_k}\right.\\
& \left. -\lim_{k\to\infty}\int_{\partial B_{l_{k+1}}} \gamma_{k+1}\,\nabla f\cdot\nu_{l_{k+1}}\,g\,\rho\,d\sigma_{l_{k+1}}\right)\\
& -\frac{1}{2}\left(\lim_{k\to-\infty}\int_{\partial B_{r_{k-1}}} \overline{\gamma}_k\,\nabla f\cdot\nu_{r_{k-1}}\,g\,\rho\,d\sigma_{r_{k-1}} + \sum_{k\in\mathbb{Z}}\int_{\partial B_{r_k}} (\overline{\gamma}_{k+1}-\overline{\gamma}_k)\,\nabla f\cdot\nu_{r_k}\,g\,\rho\,d\sigma_{r_k}\right)\\
= \quad & -\frac{1}{2}\left(\sum_{k\in\mathbb{Z}}\int_{\partial B_{l_k}} (\gamma_{k+1}-\gamma_k)\,\nabla f\cdot\nu_{l_k}\,g\,\rho\,d\sigma_{l_k} - \int_{\partial B_{m_0}} \gamma\,\nabla f\cdot\nu_{m_0}\,g\,\rho\,d\sigma_{m_0}\right)\\
& -\frac{1}{2}\left(\int_{\partial B_{m_0}} \overline{\gamma}\,\nabla f\cdot\nu_{m_0}\,g\,\rho\,d\sigma_{m_0} + \sum_{k\in\mathbb{Z}}\int_{\partial B_{r_k}} (\overline{\gamma}_{k+1}-\overline{\gamma}_k)\,\nabla f\cdot\nu_{r_k}\,g\,\rho\,d\sigma_{r_k}\right),
\end{aligned}
$$

because

$$
\lim_{k\to-\infty}\int_{\partial B_{l_{k-1}}} \gamma_k\,\nabla f\cdot\nu_{l_{k-1}}\,g\,\rho\,d\sigma_{l_{k-1}} = \lim_{k\to-\infty}\sum_{j=1}^{d}\int_{B_{l_{k-1}}} \gamma_k\,\partial_j(\partial_j f\,g\,\rho)\,dx = 0,
$$

by (HY1) and Lebesgue, since $\partial_j(\partial_j f\,g\,\rho)\in L^1_{loc}(\mathbb{R}^d)$. Similarly

$$
\begin{aligned}
& \lim_{k\to\infty}\int_{\partial B_{l_{k+1}}} \gamma_{k+1}\,\nabla f\cdot\nu_{l_{k+1}}\,g\,\rho\,d\sigma_{l_{k+1}} = \lim_{k\to\infty}\sum_{j=1}^{d}\int_{B_{l_{k+1}}} \gamma_{k+1}\,\partial_j(\partial_j f\,g\,\rho)\,dx\\
& = \sum_{j=1}^{d}\int_{B_{m_0}} \gamma\,\partial_j(\partial_j f\,g\,\rho)\,dx = \int_{\partial B_{m_0}} \gamma\,\nabla f\cdot\nu_{m_0}\,g\,\rho\,d\sigma_{m_0},
\end{aligned}
$$

and

$$
\lim_{k\to-\infty}\int_{\partial B_{r_{k-1}}} \overline{\gamma}_k\,\nabla f\cdot\nu_{r_{k-1}}\,g\,\rho\,d\sigma_{r_{k-1}} = \int_{\partial B_{m_0}} \overline{\gamma}\,\nabla f\cdot\nu_{m_0}\,g\,\rho\,d\sigma_{m_0}.
$$

□

Remark 2.2.2. *The integration by parts formula in Proposition 2.2.1 extends to $f(x)=|\|x\|-a|$, $a\in\mathbb{R}$, and the coordinate projections.*

2.3 Strict Fukushima decomposition

A positive Radon measure μ on $\mathbb{R}^d$ is said to be of *finite energy integral* if

$$\int_{\mathbb{R}^d} |f(x)|\,\mu(dx) \le C\sqrt{\mathcal{E}_1(f,f)},\ f \in D(\mathcal{E}) \cap C_0(\mathbb{R}^d),$$

where C is some constant independent of f. A positive Radon measure μ on $\mathbb{R}^d$ is of finite energy integral if and only if there exists a unique function $U_1\,\mu \in D(\mathcal{E})$ such that

$$\mathcal{E}_1(U_1\,\mu, f) = \int_{\mathbb{R}^d} f(x)\,\mu(dx),$$

for all $f \in D(\mathcal{E}) \cap C_0(\mathbb{R}^d)$. $U_1\,\mu$ is called 1-potential of μ. In particular, $R_1\mu$ is a $\rho\phi\,dx$-version of $U_1\mu$. The measures of finite energy integral are denoted by S_0.
Let further

$$S_{00} := \{\mu \in S_0 \mid \mu(\mathbb{R}^d) < \infty, \|U_1\mu\|_\infty < \infty\},$$

where $\|f\|_\infty := \inf\{c > 0 \mid \int_{\mathbb{R}^d} 1_{\{|f|>c\}}\,\rho\phi\,dx = 0\}$.

Lemma 2.3.1. *Suppose (HY1)-(HY3) hold. Then for $l \in (0,\infty)$ and $f \in C^\infty(\overline{B}_l)$,*

$$\int_{\partial B_l} |\,f\,|\,\rho\,d\sigma_l \le C(l)\left(\int_{B_l} \|\nabla f\,\|^2\,\rho\,\phi\,dx + \int_{B_l} |\,f\,|^2\,\rho\,\phi\,dx\right)^{1/2},$$

where $C : (0,\infty) \to \mathbb{R}$ is an increasing function. In particular, for any $f \in D(\mathcal{E})$

$$\int_{\partial B_l} |\,f\,|\,\rho\,d\sigma_l \le C(l)\,\|\,f\,\|_{D(\mathcal{E})}.$$

Proof. Since ∂B_l has Lipschitz boundary (actually C^∞-boundary), we can see from [25, Theorem 1 in Section 4.3] (and [68, Theorem 5.1 (i)]) that there exists a constant $\sqrt{2}$ independent of l, such that for $f \in C^\infty(\overline{B}_l)$

$$\int_{\partial B_l} |\,f\,|\,\rho\,d\sigma_l \le \sqrt{2}\int_{B_l} \big(\|\,\nabla f\,\|\,\rho\, +\, 2|\,\xi\,f\,|\,\|\nabla\xi\| + |\,f\,|\,\rho\big)\,dx$$

$$\le \sqrt{2}\left[\left(\int_{B_l} \|\nabla\,f\,\|^2\,\rho\,dx\right)^{1/2}\ \big(\rho\,dx(B_l)\big)^{1/2} + 2\left(\int_{B_l} |\,f\,|^2\,\rho\,dx\right)^{1/2}\ \left(\int_{B_l} \|\,\nabla\xi\,\|^2\,dx\right)^{1/2}\right.$$

$$+\left(\int_{B_l} |\, f\,|^2\,\rho\,dx\right)^{1/2}\ \left(\rho\,dx(B_l)\right)^{1/2}\Bigg]$$

$$\leq \sqrt{\frac{8}{\delta_l}}\Big(\ \left(\rho\,dx(B_l)\right)^{1/2}+||\nabla\xi||_{L^2(B_l,dx)}\Big)\left(\int_{B_l}\|\nabla f\,\|^2\,\rho\,\phi\,dx+\int_{B_l}|\, f\,|^2\,\rho\,\phi\,dx\right)^{1/2}.$$

This is the first statement. Let $f \in C_0^\infty(\mathbb{R}^d)$. Then, by the above, since f restricted to $\overline{B}_l$ is in $C^\infty(\overline{B}_l)$,

$$\int_{\partial B_l}|\, f\,|\,\rho\,d\sigma_l \leq C(l)\left(\int_{B_l}\|\nabla f\,\|^2\,\rho\,\phi\,dx+\int_{B_l}|\, f\,|^2\,\rho\,\phi\,dx\right)^{1/2}\leq C(l)\,||f||_{D(\mathcal{E})}.$$

Since $C_0^\infty(\mathbb{R}^d)$ is dense in $D(\mathcal{E})$, the second statement follows. □

A positive Borel measure μ on $\mathbb{R}^d$ is said to be smooth in the strict sense if there exists a sequence $(E_k)_{k\geq 1}$ of Borel sets increasing to $\mathbb{R}^d$ such that $1_{E_k}\cdot\mu \in S_{00}$ for each k and

$$\mathbb{P}_x(\lim_{k\to\infty}\sigma_{\mathbb{R}^d\setminus E_k}\geq\zeta)=1\ ,\ \ \forall x\in\mathbb{R}^d.$$

Here $\sigma_B := \inf\{t>0\,|\,X_t\in B\}$ for $B\in\mathcal{B}(\mathbb{R}^d)$. The totality of the smooth measures in the strict sense is denoted by S_1 (see [35]).

Lemma 2.3.2. *Suppose (HY1)-(HY4) hold. Let $l\in(0,\infty)$. Then, for any relatively compact open set G, $1_G\cdot\rho d\sigma_l\in S_{00}$. In particular, $\rho d\sigma_l\in S_1$.*

Proof. Let $l\in(0,\infty)$. By Lemma 2.3.1, $\rho\,d\sigma_l\in S_0$. Let us first show that $\rho\,d\sigma_l\in S_1$ with respect to an increasing sequence of open sets $(E_k)_{k\geq1}$. Choose $\varphi,\overline{\varphi}\in L_b^1(\mathbb{R}^d,\rho\phi\,dx)$, $0<\varphi,\overline{\varphi}\leq 1$ $\rho\phi\,dx$-a.e. By assumption (HY4) $R_1(\rho d\sigma_l)$ is continuous. Since furthermore $R_1\varphi$ is continuous and $R_1\varphi$ is strictly positive, it follows that

$$E_k := \{\,R_1(\rho d\sigma_l)<k^2R_1\varphi\,\},\ \ k\geq 1,$$

are open sets that increase to $\mathbb{R}^d$. Choosing the constant function $1\in C^\infty(\overline{B}_l)$ in Lemma 2.3.1 we see that $\rho d\sigma_l$ is finite. Then, clearly $1_{E_k}\cdot\rho d\sigma_l$ is also finite for all $k\geq1$. So, it remains to show that the correspondteing 1-potentials $U_1(1_{E_k}\cdot\rho d\sigma_l)$ are $\rho\phi\,dx$-essentially bounded. Let $(G_1\overline{\varphi})_{E_k}$ be the 1-reduced function of $G_1\overline{\varphi}$ on E_k as defined in [69]. Then $\mathbb{E}_\cdot\left[\int_{\sigma_{E_k}}^\infty e^{-t}\overline{\varphi}(X_t)\,dt\right]$ is a $\rho\phi\,dx$-version of $(G_1\overline{\varphi})_{E_k}$. We have (for intermediate steps see [69, p.416])

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \overline{\varphi}\, U_1\left(1_{E_k}\cdot \rho d\sigma_l\right)\rho\phi\, dx = \int_{\partial B_l} R_1\overline{\varphi}\ \ 1_{E_k}\cdot\rho d\sigma_l \\
&= \int_{\partial B_l} \mathbb{E}.\left[\int_{\sigma_{E_k}}^{\infty} e^{-t}\overline{\varphi}(X_t)\,dt\right] 1_{E_k}\cdot\rho d\sigma_l = \mathcal{E}_1\left(\left(G_1\overline{\varphi}\right)_{E_k}, U_1(1_{E_k}\cdot\rho d\sigma_l)\right) \\
&= \mathcal{E}_1\left(\left(G_1\overline{\varphi}\right)_{E_k}, U_1(1_{E_k}\cdot\rho d\sigma_l)\wedge k^2 G_1\varphi\right) \\
&\leq \mathcal{E}_1\left(G_1\overline{\varphi}, U_1(1_{E_k}\cdot\rho d\sigma_l)\wedge k^2 G_1\varphi\right) = \int_{\mathbb{R}^d}\overline{\varphi}\left(U_1(1_{E_k}\cdot\rho d\sigma_l)\wedge k^2 G_1\varphi\right)\ \rho\phi\, dx.
\end{aligned}
$$

This implies that $U_1(1_{E_k}\cdot\rho d\sigma_l)\leq k^2\ \ \rho\phi\, dx$-a.e. Hence, $1_{E_k}\cdot\rho d\sigma_l\in S_{00}$ for all $k\geq 1$. Since moreover $\mathbb{P}_x(\lim_{k\to\infty}\sigma_{\mathbb{R}^d\setminus E_k}\geq\zeta)=1,\ \forall x\in\mathbb{R}^d$ is easily deduced from the absolute continuity condition, we finally obtain $\rho d\sigma_l\in S_1$ with respect to $(E_k)_{k\geq 1}$.
For a relatively compact open set G, we know that there exists $k_0\in\mathbb{N}$ with $G\subset\overline{G}\subset E_{k_0}$. Hence, $U_1\left(1_G\cdot\rho\, d\sigma_l\right)\leq U_1(1_{E_{k_0}}\cdot\rho\, d\sigma_l)\leq k_0^2\, G_1\varphi\leq k_0^2$. Therefore, $1_G\cdot\rho d\sigma_l\in S_{00}$. □

By Lemma 2.3.2, we know that $\rho d\sigma_r\in S_1$ for any $r\in(0,\infty)$. Hence, by [35, Theorem 5.1.7] there exists a unique $(\bar{\ell}^r_t)_{t\geq 0}\in A^+_{c,1}$ with Revuz measure $\rho d\sigma_r$, such that

$$
\mathbb{E}_x\left[\int_0^\infty e^{-t}\, d\bar{\ell}^r_t\right] = R_1(\rho d\sigma_r)(x)\,,\ \ \forall x\in\mathbb{R}^d.
$$

Here, $A^+_{c,1}$ denotes the positive continuous additive functionals in the strict sense.

Theorem 2.3.3. *Suppose (HY1)-(HY4) hold. Then, for any relatively compact open set G, $1_G\cdot\mu\in S_{00}-S_{00}$, where*

$$
\mu=\sum_{k\in\mathbb{Z}}\left(\frac{\gamma_{k+1}-\gamma_k}{2}\,\rho\, d\sigma_{l_k}+\frac{\overline{\gamma}_{k+1}-\overline{\gamma}_k}{2}\,\rho\, d\sigma_{r_k}\right)+\frac{\overline{\gamma}-\gamma}{2}\,\rho\, d\sigma_{m_0}.
$$

In particular $\mu\in S_1-S_1$.

Proof. It suffices to show that $\mu_n\in S_{00}$ for any $n\in\mathbb{N},\ n>m_0$, where

$$
\mu_n := 1_G\cdot\left(\sum_{k\in\mathbb{Z}}|\,\gamma_{k+1}-\gamma_k\,|\,\rho\, d\sigma_{l_k}+\sum_{\{k\in\mathbb{Z}:r_k<n\}}|\,\overline{\gamma}_{k+1}-\overline{\gamma}_k\,|\,\rho\, d\sigma_{r_k}+|\,\overline{\gamma}-\gamma\,|\,\rho\, d\sigma_{m_0}\right).
$$

First we show that $\mu_n \in S_0$. Let $f \in C_0^\infty(\mathbb{R}^d)$. Then, by Lemma 2.3.1

$$\int_{\mathbb{R}^d} |\, f \,|\, d\mu_n \le C(n) \left(\sum_{k\in\mathbb{Z}} |\, \gamma_{k+1} - \gamma_k \,| + \sum_{\{k\in\mathbb{Z}: r_k < n\}} |\, \overline{\gamma}_{k+1} - \overline{\gamma}_k \,| + |\, \overline{\gamma} - \gamma \,| \right) \, ||f||_{D(\mathcal{E})},$$

where $C(n)$ is as in Lemma 2.3.1. Now, we show $\mu_n \in S_{00}$. Let $f \in C_0^\infty(B_n)$ such that $f = 1$ on $\overline{B}_{n-\varepsilon}$ where $\varepsilon > 0$ is small enough to satisfy $(n-\varepsilon) > max_{\{k\in\mathbb{Z}: r_k<n\}} r_k$. Then, by Lemma 2.3.1 again

$$\begin{aligned} \mu_n(\mathbb{R}^d) &\le C(n) \left(\sum_{k\in\mathbb{Z}} |\, \gamma_{k+1} - \gamma_k \,| + \sum_{\{k\in\mathbb{Z}: r_k < n\}} |\, \overline{\gamma}_{k+1} - \overline{\gamma}_k \,| + |\, \overline{\gamma} - \gamma \,| \right) \left(\int_{B_n} \rho\phi \, dx \right)^{1/2} \\ &< \infty. \end{aligned}$$

By the proof of Lemma 2.3.2, we can see that

$$U_1 \left(1_{E_k} \cdot \rho \, d\sigma_l \right) \le k^2, \;\; k \ge 1,$$

independently of $l \in (0,\infty)$. For any relatively compact open set G, there exists $k_0 \in \mathbb{N}$ such that $G \subset \overline{G} \subset E_{k_0}$. Since $U_1 \left(1_G \cdot \rho \, d\sigma_l \right) \le U_1 (1_{E_{k_0}} \cdot \rho \, d\sigma_l)$ for any l, we obtain for $\rho\phi \, dx$-a.e. $x \in \mathbb{R}^d$

$$\begin{aligned} U_1\mu_n(x) &\le \sum_{k\in\mathbb{Z}} |\, \gamma_{k+1} - \gamma_k \,|\, U_1 \left(1_G \cdot \rho \, d\sigma_{l_k} \right)(x) \\ &+ \sum_{\{k\in\mathbb{Z}: r_k < n\}} |\, \overline{\gamma}_{k+1} - \overline{\gamma}_k \,|\, U_1 \left(1_G \cdot \rho \, d\sigma_{r_k} \right)(x) + |\, \overline{\gamma} - \gamma \,|\, U_1 \left(1_G \cdot \rho \, d\sigma_{m_0} \right)(x) \\ &\le k_0^2 \left(\sum_{k\in\mathbb{Z}} |\, \gamma_{k+1} - \gamma_k \,| + \sum_{\{k\in\mathbb{Z}: r_k < n\}} |\, \overline{\gamma}_{k+1} - \overline{\gamma}_k \,| + |\, \overline{\gamma} - \gamma \,| \right) < \infty. \end{aligned}$$

Therefore, $\mu \in S_{00}$. □

Remark 2.3.4. *Let $E_k, k \ge 1$, be open sets such that $E_k \nearrow \mathbb{R}^d$ and let $\mu = \mu_A, \mu_n = \mu_{A^n} \in S_1$ w.r.t. $(E_k)_{k\ge1}$, $A, A^n \in A_{c,1}^+$, $n \ge 1$. If $\mu_A = \sum_{n\ge1} \mu_{A^n}$, then $A = \sum_{n\ge1} A^n$, since $R_1(f d\mu_A)(x) = \sum_{n\ge1} R_1(f d\mu_{A^n})(x)$ for any $x \in \mathbb{R}^d$, $f \in C_0(\mathbb{R}^d)$.*

Theorem 2.3.5. *Suppose (HY1)-(HY4) hold. For any relatively compact open set G* *and* $j = 1, \dots, d$,

$$1_G \cdot \partial_j \rho \, \phi \, dx \in S_{00} - S_{00}.$$

In particular $\partial_j \rho \, \phi \, dx \in S_1 - S_1$, $j = 1, \dots, d$.

Proof. It suffices to show that $1_G \cdot |\partial_j \rho| \, \phi \, dx \in S_{00}$. By Remark 2.1.2, it is easy to see that $1_G \cdot |\partial_j \rho| \, \phi \, dx \in S_0$ and that $1_G \cdot |\partial_j \rho| \, \phi \, dx(\mathbb{R}^d) < \infty$. We can show that

$$\|U_1(1_G \cdot |\partial_j \rho| \, \phi \, dx)\|_\infty < \infty$$

by replacing the E_k in the proof of Lemma 2.3.2 with $E'_k := \{ R_1(1_G \cdot |\partial_j \rho| \, \phi \, dx) < k^2 R_1 \varphi \}$. $\square$

2.4 Proof of the main theorem

Proof of Theorem 2.1.6. (i) Applying [35, Theorem 5.5.5] to $(\mathcal{E}, D(\mathcal{E}))$ and to the coordinate projections which are in $D(\mathcal{E})_{b,loc}$, the identification of the martingale part as Brownian motion is easy. Concerning the drift part the strict decomposition holds true by Lemma 2.3.2, Remark 2.2.2, Theorem 2.3.3, Remark 2.3.4 and Theorem 2.3.5. Note that equation (2.4) holds for all $t \geq 0$ by (2.3).
(ii) Let $f(x) := \|x\|$, $x \in \mathbb{R}^d$. Then $\partial_j f$ is everywhere bounded by one (except in zero). By Remark 2.2.2 for $f(x) = \|x\|$, $g \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned}
& -\mathcal{E}(f,g) \\
= \; & \int_{\mathbb{R}^d} \left(\frac{1}{2} \Delta f + \nabla f \cdot \frac{\nabla \rho}{2\rho} \right) g \, \rho \, \phi \, dx + \frac{\overline{\gamma} - \gamma}{2} \int_{\partial B_{m_0}} \nabla f \cdot \nu_{m_0} \, g \, \rho \, d\sigma_{m_0} \\
& + \sum_{k \in \mathbb{Z}} \left(\frac{\gamma_{k+1} - \gamma_k}{2} \int_{\partial B_{l_k}} \nabla f \cdot \nu_{l_k} \, g \, \rho \, d\sigma_{l_k} + \frac{\overline{\gamma}_{k+1} - \overline{\gamma}_k}{2} \int_{\partial B_{r_k}} \nabla f \cdot \nu_{r_k} \, g \, \rho \, d\sigma_{r_k} \right) \\
= \; & \int_{\mathbb{R}^d} \left(\frac{x}{\|x\|} \cdot \frac{\nabla \rho}{2\rho} \right) g \, \rho \, \phi \, dx + \frac{\overline{\gamma} - \gamma}{2} \int_{\partial B_{m_0}} g \, \rho \, d\sigma_{m_0} \\
& + \sum_{k \in \mathbb{Z}} \left(\frac{\gamma_{k+1} - \gamma_k}{2} \int_{\partial B_{l_k}} g \, \rho \, d\sigma_{l_k} + \frac{\overline{\gamma}_{k+1} - \overline{\gamma}_k}{2} \int_{\partial B_{r_k}} g \, \rho \, d\sigma_{r_k} \right).
\end{aligned}$$

Note that $\|\nabla f\|^2 = 1$. Thus applying [35, Theorem 5.5.5] to f, which is in $D(\mathcal{E})_{b,loc}$, we obtain similarly to (i)

$$\|X_t\| = \|x\| + B_t + \int_0^t \frac{X_s}{\|X_s\|} \cdot \frac{\nabla\rho}{2\rho}(X_s)\, ds + \overline{N}_t, \tag{2.5}$$

$\mathbb{P}_x$ -a.s. for any $x \in \mathbb{R}^d$, $t \geq 0$, where B is a standard one dimensional Brownian motion starting from zero and

$$\overline{N}_t := \sum_{k\in\mathbb{Z}} \left(\frac{\gamma_{k+1} - \gamma_k}{\gamma_{k+1} + \gamma_k}\ \ell_t^{l_k} + \frac{\overline{\gamma}_{k+1} - \overline{\gamma}_k}{\overline{\gamma}_{k+1} + \overline{\gamma}_k}\ \ell_t^{r_k} \right) + \frac{\overline{\gamma} - \gamma}{\overline{\gamma} + \gamma}\ \ell_t^{m_0}.$$

Therefore, the first statement follows. In particular, we may apply the symmetric Itô-Tanaka formula (see [54, VI. (1.25)]) and obtain

$$\big|\|X_t\| - a\big| = \big|\|x\| - a\big| + \int_0^t sign(\|X_s\| - a)d\|X_s\| + \ell_t^a(\|X\|), \tag{2.6}$$

$\mathbb{P}_x$ -a.s. for any $x \in \mathbb{R}^d$, $t \geq 0$, where $\ell_t^a(\|X\|)$ is the symmetric semimartingale local time of $\|X\|$ at $a \in (0,\infty)$ as defined in [54, VI.(1.25)] and $sign(x) = 1$ for $x > 0$, -1 for $x < 0$ and 0 for $x = 0$. Let $h(x) := \big|\|x\| - a\big|$, $a \in \{m_0, l_k, r_k : k \in \mathbb{Z}\}, x \in \mathbb{R}^d$. Then $\partial_j h$ is everywhere bounded by one (except in a). Note that for $x \neq 0$, $\|x\| \neq a$ $\Delta h(x) = 0$ and

$$\nabla h(x) = \frac{x}{\|x\|} 1_{\{\|x\|>a\}} - \frac{x}{\|x\|} 1_{\{\|x\|<a\}}.$$

Let, for instance, $a = l_r$, $r \in \mathbb{Z}$. By Remark 2.2.2 we obtain for $h(x) := \big|\|x\| - l_r\big|$, $g \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned}
& -\mathcal{E}(h,g) \\
= \ & \int_{\mathbb{R}^d} \left(\frac{1}{2}\Delta h + \nabla h \cdot \frac{\nabla\rho}{2\rho} \right) g\, \rho\, \phi\, dx + \frac{\overline{\gamma} - \gamma}{2} \int_{\partial B_{m_0}} \nabla h \cdot \nu_{m_0}\, g\, \rho\, d\sigma_{m_0} \\
& + \sum_{\substack{k\in\mathbb{Z},\\ k\neq r}} \left(\frac{\gamma_{k+1} - \gamma_k}{2} \int_{\partial B_{l_k}} \nabla h \cdot \nu_{l_k}\, g\, \rho\, d\sigma_{l_k} \right) \\
& + \sum_{k\in\mathbb{Z}} \left(\frac{\overline{\gamma}_{k+1} - \overline{\gamma}_k}{2} \int_{\partial B_{r_k}} \nabla h \cdot \nu_{r_k}\, g\, \rho\, d\sigma_{r_k} \right) + \frac{\gamma_{r+1} + \gamma_r}{2} \int_{\partial B_{l_r}} \frac{x}{\|x\|} \cdot \nu_{l_r}\, g\, \rho\, d\sigma_{l_r}
\end{aligned}$$

$$
\begin{aligned}
= \quad & \int_{\mathbb{R}^d} \left(\frac{x}{\|x\|} \cdot \frac{\nabla \rho}{2\rho} \right) (1_{\{\|x\|>l_r\}} - 1_{\{\|x\|<l_r\}}) g\, \rho\, \phi\, dx \\
& + \frac{\overline{\gamma} - \gamma}{2} \int_{\partial B_{m_0}} (1_{\{\|x\|>l_r\}} - 1_{\{\|x\|<l_r\}}) g\, \rho\, d\sigma_{m_0} \\
& + \sum_{\substack{k \in \mathbb{Z}, \\ k \neq r}} \left(\frac{\gamma_{k+1} - \gamma_k}{2} \int_{\partial B_{l_k}} (1_{\{\|x\|>l_r\}} - 1_{\{\|x\|<l_r\}}) g\, \rho\, d\sigma_{l_k} \right) \\
& + \sum_{k \in \mathbb{Z}} \left(\frac{\overline{\gamma}_{k+1} - \overline{\gamma}_k}{2} \int_{\partial B_{r_k}} (1_{\{\|x\|>l_r\}} - 1_{\{\|x\|<l_r\}}) g\, \rho\, d\sigma_{r_k} \right) \\
& + \frac{\gamma_{r+1} + \gamma_r}{2} \int_{\partial B_{l_r}} g\, \rho\, d\sigma_{l_r}.
\end{aligned}
$$

Thus, applying [35, Theorem 5.5.5] to h, which is in $D(\mathcal{E})_{b,loc}$ and using (2.5), we obtain again similarly to (i)

$$
\big|\|X_t\| - a\big| = \big|\|x\| - a\big| + \int_0^t sign(\|X_s\| - a) d\|X_s\| + \ell_t^a, \tag{2.7}
$$

$\mathbb{P}_x$ -a.s. for any $x \in \mathbb{R}^d$, $t \geq 0$. Comparing (2.6)and (2.7), we get the result.

□

Remark 2.4.1. *(see [35, Theorem 4.7.1 (i), (iii), and Exercise 4.7.1]) If $(\mathcal{E}, D(\mathcal{E}))$ is irreducible, then for any nearly Borel non-exceptional set B,*

$$
\mathbb{P}_x(\sigma_B < \infty) > 0, \quad \forall x \in \mathbb{R}^d.
$$

If $(\mathcal{E}, D(\mathcal{E}))$ is irreducible and recurrent, then for any nearly Borel non-exceptional set B,

$$
\mathbb{P}_x(\sigma_B \circ \theta_n < \infty, \forall n \geq 0) = 1, \quad \forall x \in \mathbb{R}^d.
$$

Here $(\theta_t)_{t \geq 0}$ is the shift operator. Moreover, in this case any excessive function is constant. In particular, the ergodic Theorem [35, Theorem 4.7.3 (iv)] holds. A sufficient condition for recurrence is given by

$$
\int_1^\infty \frac{r}{\rho\phi\, dx(B_r)} dr = \infty,
$$

see [61, Theorem 3].

Chapter 3

Symmetric distorted Brownian motion

In this chapter in order to simplify notation while handling inequalities or estimates we make the convention that unless otherwise specified $c > 0$ stands for an arbitrary constant whose value may vary from inequality to inequality. We will refer to [35] till the end of this chapter, hence some of its standard notations may be adopted below without definition.

3.1 Preliminaries and the absolute continuity condition

In this section we denote E by a locally compact separable metric space.

3.1.1 Global setting

Throughout this section, we let $(\mathcal{E}, D(\mathcal{E}))$ be a symmetric, strongly local, regular Dirichlet form on $L^2(E, m)$ where m is a positive Radon measure on $(E, \mathcal{B}(E))$ with full support on E. We further assume throughout this section that $\mathcal{E}$ admits a carré du champ

$$\Gamma : D(\mathcal{E}) \times D(\mathcal{E}) \to L^1(E, m)$$

as in [18, Definition 4.1.2]. As usual we define $\mathcal{E}_1(f, g) := \mathcal{E}(f, g) + (f, g)_{L^2(E,m)}$ for $f, g \in D(\mathcal{E})$ and $\| f \|_{D(\mathcal{E})} := \mathcal{E}_1(f, f)^{1/2}$, $f \in D(\mathcal{E})$. Let $(T_t)_{t>0}$ and $(G_\alpha)_{\alpha>0}$ be the $L^2(E, m)$-semigroup and resolvent associated to $(\mathcal{E}, D(\mathcal{E}))$ and $(L, D(L))$ be the corresponding generator (see [44, Diagram 3, p. 39]). Let Cap be the capacity related to the regular symmetric Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ as defined in [35, 2.1]. We say that a function f is locally in $D(\mathcal{E})_b$ ($f \in D(\mathcal{E})_{b,loc}$ in notation) if for any relatively compact open set $G \subset E$, there exists a function $g \in D(\mathcal{E})_b$ such that $f = g$ m-a.e. on G. We consider the condition

(**H1**) There exists a $\mathcal{B}(E) \times \mathcal{B}(E)$ measurable non-negative map $p_t(x, y)$ such that

$$P_t f(x) := \int_E p_t(x, y)\, f(y)\, m(dy)\,,\ t > 0, \quad x \in E, \quad f \in \mathcal{B}_b(E),$$

is a (temporally homogeneous) sub-Markovian transition function (see [22, 1.2]) and an m-version of $T_t f$ if $f \in L^2(E, m)_b$.

$p_t(x, y)$ is called the transition kernel density or heat kernel. Taking the Laplace transform of $p_\cdot(x, y)$, we see that (**H1**) implies that there exists a $\mathcal{B}(E) \times \mathcal{B}(E)$ measurable non-negative map $r_\alpha(x, y)$ such that

$$R_\alpha f(x) := \int_E r_\alpha(x, y)\, f(y)\, m(dy)\,,\ \alpha > 0,\ x \in E, f \in \mathcal{B}_b(E), \tag{3.1}$$

is an m-version of $G_\alpha f$ if $f \in L^2(E, m)_b$. $r_\alpha(x, y)$ is called the resolvent kernel density. For a signed Radon measure μ on E, let us define

$$R_\alpha \mu(x) = \int_E r_\alpha(x, y)\, \mu(dy)\,, \quad \alpha > 0, \quad x \in E, \tag{3.2}$$

whenever this makes sense. Throughout this chapter, we set $P_0 := id$. Furthermore, assuming that (**H1**) holds, we can consider the condition

(**H2**) There exists a Hunt process with transition function $(P_t)_{t\geq 0}$.

We recall that (**H2**) means that there exists a Hunt process

$$\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \zeta, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E_\Delta}), \tag{3.3}$$

with state space E and lifetime ζ such that $P_t(x, B) := P_t 1_B(x) = \mathbb{P}_x(X_t \in B)$ for any $x \in E$, $B \in \mathcal{B}(E)$, $t \geq 0$. Here, Δ is the cemetery point and as usual any function $f : E \to \mathbb{R}$ is extended to $\{\Delta\}$ by setting $f(\Delta) := 0$. $E_\Delta := E \cup \{\Delta\}$ is the one-point compactification if E is not already compact, if E is compact then Δ is added to E as an isolated point.

Remark 3.1.1. *Note that if* (**H1**) *and* (**H2**) *hold, then* $\mathbb{M}$ *is associated with* $(\mathcal{E}, D(\mathcal{E}))$ *and satisfies the absolute continuity condition as stated in [35, p. 165].*

Below, we present two methods to obtain $\mathbb{M}$ as in Remark 3.1.1.

The Feller semigroup method

Assuming (**H1**), a Hunt process as in (**H2**) can be constructed by means of a Feller semigroup (cf. [15, Chapter I, (9.4) Theorem]). For the definition of Feller semigroup, we refer to [22, Section 2.2].

Remark 3.1.2. *Under* (**H1**), $(P_t)_{t\geq 0}$ *is a Feller semigroup, if*

(i) $\forall f \in C_\infty(E)$, $\lim_{t\to 0} P_t f = f$ *uniformly on* E.

(ii) $P_t C_\infty(E) \subset C_\infty(E)$ *for each* $t > 0$.

It is well known that the condition of uniform convergence in Remark 3.1.2 (i) can be relaxed to pointwise convergence (see for instance [22, Section 2.2, Exercise 4]). The conditions of Remark 3.1.2 can even be further relaxed to the conditions of the following lemma which are suitable for us and we add the proof for the convenience of the reader.

Lemma 3.1.3. *Suppose that* (**H1**) *holds and that*

(i) $\lim_{t\to 0} P_t f(x) = f(x)$ *for each* $x \in E$ *and* $f \in C_0(E)$.

(ii) $P_t C_0(E) \subset C_\infty(E)$ *for each* $t > 0$.

Then $(P_t)_{t\geq 0}$ *is a Feller semigroup. In particular* (***H2***) *holds.*

Proof. Let $t > 0$, $\alpha > 0, x \in E$ and $f \in C_0(E)$. Using (**H1**), we obtain

$$P_t R_\alpha f(x) = \int_0^\infty e^{-\alpha s} P_{t+s} f(x)\, ds = e^{\alpha t} \int_t^\infty e^{-\alpha s} P_s f(x)\, ds,$$

and so

$$\begin{aligned} \Big| P_t R_\alpha f(x) - R_\alpha f(x) \Big| &= \Big| e^{\alpha t} \int_t^\infty e^{-\alpha s} P_s f(x)\, ds - \int_0^\infty e^{-\alpha s} P_s f(x)\, ds \Big| \\ &\leq (e^{\alpha t} - 1) \int_t^\infty e^{-\alpha s} P_s |f|(x)\, ds + \int_0^t e^{-\alpha s} P_s |f|(x)\, ds. \end{aligned}$$

The first term of the last expression is bounded by $(e^{\alpha t}-1)\|f\|_\infty \cdot \frac{1}{\alpha}$ and the second by $t\|f\|_\infty$. Hence, $P_tR_\alpha f$ converges to $R_\alpha f$ uniformly as $t \to 0$ for any $f \in C_0(E)$.
Let $D := \{R_\alpha f \mid f \in C_0(E)\}$ and $D_{const} := \{f + c \mid f \in D,\ c \in \mathbb{R}\}$. Then clearly $D_{const} \subset C(E_\Delta)$ by (ii) and D_{const} is a linear space. In order to show that $D \subset C_\infty(E)$ densely (w.r.t. the sup-norm) it suffices to show that $D_{const} \subset C(E_\Delta)$ densely (w.r.t. the sup-norm). The dual space of $C(E_\Delta)$ is the space of bounded signed Radon measures on E_Δ that we denote by A. Let $\mu \in A$ be such that

$$\int_{E_\Delta} h\, d\mu = 0, \qquad \forall h \in D_{const}. \tag{3.4}$$

Note that $\alpha R_\alpha f$ converges pointwise and boundedly to $f \in C_0(E)$ as $\alpha \to \infty$ by (i). Hence by Lebesgue and (3.4), for all $f \in C_0(E)$ and $c \in \mathbb{R}$.

$$\int_{E_\Delta} (f + c)\, d\mu = \lim_{\alpha\to\infty} \int_{E_\Delta} (\alpha R_\alpha f + c)\, d\mu = 0.$$

Since $C_0(E) \subset C_\infty(E)$ densely and $C(E_\Delta) = \{f + c \mid f \in C_\infty(E),\ c \in \mathbb{R}\}$, we obtain $\mu = 0$. Therefore $D_{const} \subset C(E_\Delta)$ densely by the Hahn-Banach theorem. Let $g \in C_\infty(E)$ and $\varepsilon > 0$. By the above, we can choose $f \in C_0(E)$ with $\|g - R_\alpha f\|_\infty < \varepsilon$. Since

$$\|P_tg - g\|_\infty \le \|P_t(g - R_\alpha f)\|_\infty + \|P_tR_\alpha f - R_\alpha f\|_\infty + \|R_\alpha f - g\|_\infty,$$

P_tg converges to g uniformly on E as $t \to 0$ for any $g \in C_\infty(E)$. Thus Remark 3.1.2 (i) holds. Let $(g_n)_{n\ge0} \subset C_0(E)$ converge uniformly to $g \in C_\infty(E)$. Then since

$$\|P_tg - P_tg_n\|_\infty \le \|g - g_n\|_\infty,$$

P_tg_n converges to P_tg uniformly as $n \to \infty$. Therefore, $P_tg \in C_\infty(E)$ and Remark 3.1.2 (ii) is shown. □

Remark 3.1.4. *We will see that one can use heat kernel estimates for $p_t(x, y)$ to check the assumption of Lemma 3.1.3 (i), (ii) (see Lemma 3.2.6 (i) below).*

The Dirichlet form method

The second method to obtain a Hunt process as in Remark 3.1.1, given a transition function as in (**H1**), is by the method introduced in [6, Section 4]. We shall call it

the Dirichlet form method and refine it as follows in a frame that is suitable for our purposes. We assume hence (**H1**) to hold and explain the main steps of the method and of our refinement.
Given the transition function $(P_t)_{t\geq 0}$ on E, restricted to the positive dyadic rationals $S := \bigcup_{n\in\mathbb{N}} S_n$, $S_n := \{k2^{-n} \mid k \in \mathbb{N}\cup\{0\}\}$, we construct a Markov process

$$\mathbb{M}^0 = (\Omega, \mathcal{F}^0, (\mathcal{F}^0_s)_{s\in S}, (X^0_s)_{s\in S}, (\mathbb{P}_x)_{x\in E_\Delta})$$

with transition function on E_Δ

$$P^\Delta_t(x, dy) = \begin{cases} \big[1 - P_t(x, E)\big]\delta_\Delta(dy) + P_t(x, dy), & \text{if } x \in E \\ \delta_\Delta(dy), & \text{if } x = \Delta \end{cases}$$

by Kolmogorov's method (see [54, Chapter III]). Here $\Omega := (E_\triangle)^S$ is equipped with the product σ-field $\mathcal{F}^0$, $X^0_s : (E_\triangle)^S \to E_\triangle$ are coordinate maps and $\mathcal{F}^0_s := \sigma(X^0_r \mid r \in S, r \leq s)$. By the theory of Dirichlet forms there exists a Hunt process

$$\tilde{\mathbb{M}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\zeta}, (\tilde{X}_t)_{t\geq 0}, (\tilde{\mathbb{P}}_x)_{x\in E_\Delta})$$

associated with $(\mathcal{E}, D(\mathcal{E}))$, where $\tilde{\Omega} = \{\omega = (\omega(t))_{t\geq 0} \in C([0,\infty), E_\Delta) \mid \omega(t) = \Delta,\ \forall t \geq \tilde{\zeta}\}$ (see [35, Theorem 4.5.3]). Let $\nu := gdm$, where $g \in L^1(E, m)$, $g > 0$ m-a.e., $\int_E g\ dm = 1$, and set

$$\tilde{\mathbb{P}}_\nu(\cdot) := \int_E \tilde{\mathbb{P}}_x(\cdot)\ g(x)\ m(dx).$$

Consider the one-to-one map $G : \tilde{\Omega} \to \Omega$ defined by

$$G(\omega) = \omega|_S.$$

Then G is $\tilde{\mathcal{F}}^0/\mathcal{F}^0$ measurable and $\tilde{\Omega} \in \tilde{\mathcal{F}}^0$, where $\tilde{\mathcal{F}}^0 := \sigma(\tilde{X}_s \mid s \in S)$ and exactly as in [6, Lemma 4.2 and 4.3] we can show that $\tilde{\mathbb{P}}_\nu|_{\tilde{\mathcal{F}}^0} \circ G^{-1} = \mathbb{P}_\nu$, $G(\tilde{\Omega}) \in \mathcal{F}^0$ and $\mathbb{P}_\nu(G(\tilde{\Omega})) = 1$. Then, we show [6, Lemma 4.4] with $A = G(\tilde{\Omega})$ $\forall x \in E$, i.e. if

$$\Omega_1 := \bigcap_{s>0, s\in S} \theta_s^{-1}(G(\tilde{\Omega})),$$

where $\theta_s : \Omega \to \Omega$, $\theta_s(\omega) := \omega(\cdot + s)$, for $s \in S$, is the usual shift operator, then

$$\mathbb{P}_x(\Omega_1) = 1 \tag{3.5}$$

for all $x \in E$. Actually, there is a minor inconsistency in the proof of [6, Lemma 4.4] since the wrong transition function P_s, instead of P_s^{Δ} is applied. However the Lemma remains true for $A = G(\tilde{\Omega})$ and there is no further influence on any other result in [6]. Before we go on with our refinement of the Dirichlet form method it is convenient to introduce some definitions and lemmas:
It is well-known that $T_t,\ t > 0$, restricted to $L^1(E,m) \cap L^\infty(E,m)$ can be extended to a C_0-semigroup of sub-Markovian contractions on $L^r(E,m)$ for any $r \geq 1$. We denote the corresponding generators by $(L_r, D(L_r))$ (for details we refer to [23, Lemmas 1.11 and 1.12 of Appendix B] and references therein).

Lemma 3.1.5. *Let $u \in D(L)_0 \cap \mathcal{B}_b(E)$. Then:*

(i) $\operatorname{supp}(Lu) \subset \operatorname{supp}(u)$.

(ii) It holds $u, u^2 \in D(L_1)$ and

$$L_1 u^2 = \Gamma(u,u) + 2uLu.$$

(iii) If $\Gamma(u,u) \in L^p(E,m)$ for some $p \in [2,\infty]$, then $u^2 \in D(L)_0 \cap \mathcal{B}_b(E)$.

Proof. (i) The statement follows easily from the local property of $(\mathcal{E}, D(\mathcal{E}))$, since

$$\int Lu \cdot \mathrm{v}\, dm = -\mathcal{E}(u,\mathrm{v}) = 0 \quad \forall \mathrm{v} \in D(\mathcal{E}) \text{ with } \operatorname{supp}(\mathrm{v}) \subset \mathbb{R}^d \setminus \operatorname{supp}(u).$$

(ii) Since $L^2(E,m)_0 \subset L^1(E,m)_0$, we conclude with the help of (i) that $u, Lu \in L^1(E,m)_0$. Hence $u \in D(L_1) \cap \mathcal{B}_b(E)$ by [23, Lemmas 1.11, 1.12 of Appendix B]. By [18, I. Theorem 4.2.1], it then holds $u^2 \in D(L_1) \cap \mathcal{B}_b(E)$ and

$$L_1 u^2 = \Gamma(u,u) + 2uLu.$$

(iii) By [69, Lemma 3.8 (iii)] we find $\operatorname{supp}(\,\Gamma(u,u)) \subset \operatorname{supp}(u)$ since $1_{\mathbb{R}^d \setminus \operatorname{supp}(u)} \Gamma(u,u) dm = 0$. Therefore $\Gamma(u,u) \in L^2(E,m)_0$ and so $L_1 u^2 \in L^2(E,m)$ by *(ii)*. Since $u^2 \in L^2(E,m)$ and $u^2 \in D(L_1)$ by (ii) it follows again from [23, Lemmas 1.11, 1.12 of Appendix B] that $u^2 \in D(L)$. □

Lemma 3.1.6. *Let $f \in \mathcal{B}(E)$ such that $R_1|f|$ is finite on E (for instance if $R_1|f|$ is continuous or if $f \in L^\infty(E,m)$). Then for any $t \geq 0$*

$$\lim_{\substack{s \downarrow t \\ s \in S}} P_s R_1 f(x) = e^t \int_t^\infty e^{-u}\ P_u f(x)\ du = P_t R_1 f(x).$$

In particular

$$\lim_{\substack{s \downarrow 0 \\ s \in S}} P_s R_1 f(x) = R_1 f(x) \quad \textit{for any} \;\; x \in E.$$

Proof. First note that for any function $f \in \mathcal{B}^+(E)$, we have $P_s f(x) = P_s^\Delta f(x)$ if $x \in E$. Using this, for any $f \in \mathcal{B}^+(E)$ and $x \in E$, we then obtain with Fubini

$$P_s R_1 f(x) = P_s^\Delta R_1 f(x) = \mathbb{E}_x[R_1 f(X_s^0)] = e^s \int_s^\infty e^{-u}\ P_u f(x)\ du, \quad s > 0, \tag{3.6}$$

where $\mathbb{E}_x$ denotes the expectation w.r.t. $\mathbb{P}_x$. The r.h.s. of (3.6) converges in $\mathbb{R}$ to $e^t \int_t^\infty e^{-u}\ P_u f(x)\ du$ as $s \downarrow t$, $t \geq 0$ if $R_1 f(x)$ is finite. If $R_1|f|$ is finite, then $R_1(f^+)$ as well as $R_1(f^-)$ are finite and so the assertion follows. □

Ω_1 defined in (3.5) consists of paths in Ω which have unique continuous extensions to $(0,\infty)$ which still lie in E_Δ and stay in Δ once they have hit Δ. Following the main idea of [6], we have to handle the limits at $s = 0$. This can be done assuming the following condition

(**H2**)′ We can find $\{u_n \mid n \geq 1\} \subset D(L) \cap C_0(E)$ satisfying:

(i) For all $\varepsilon \in \mathbb{Q} \cap (0,1)$ and $y \in D$, where D is any given countable dense set in E, there exists $n \in \mathbb{N}$ such that $u_n(z) \geq 1$, for all $z \in \overline{B}_{\frac{\varepsilon}{4}}(y)$ and $u_n \equiv 0$ on $E \setminus B_{\frac{\varepsilon}{2}}(y)$.

(ii) $R_1\big([(1-L)u_n]^+\big)$, $R_1\big([(1-L)u_n]^-\big)$, $R_1\big([(1-L_1)u_n^2]^+\big)$, $R_1\big([(1-L_1)u_n^2]^-\big)$ are continuous on E for all $n \geq 1$.

(iii) $R_1 C_0(E) \subset C(E)$.

(iv) For any $f \in C_0(E)$ and $x \in E$, the map $t \mapsto P_t f(x)$ is right-continuous on $(0,\infty)$.

Remark 3.1.7. *(i) By Lemma 3.1.5 (ii), $u_n^2 \in D(L_1)$ $\forall n \geq 1$. Thus $L_1 u_n^2$ in (**H2**)′ (ii) is well-defined.*

*(ii) In view of Lemma 3.1.5 (**H2**)$'$ (ii)-(iii) can be replaced by the following (stronger) condition:*

$\exists r \in [1,\infty]$ such that $R_1\big(L^r(E,m)_0\big) \subset C(E)$ and $Lu_n \in L^r(E,m)$ for any $n \geq 1$ and if $r \neq 1$, then $\Gamma(u_n,u_n)^{1/2} \in L^\infty(E,m)$, $\forall n \geq 1$.

Define

$$\Omega_0 := \{\omega \in \Omega_1 \mid \lim_{s \downarrow 0} X_s^0(\omega) \text{ exists in } \ E\}.$$

Lemma 3.1.8. *Under (**H1**) and **(H2)**$'$, we have*

$$\lim_{\substack{s \downarrow 0 \\ s \in S}} X_s^0 = x \quad \mathbb{P}_x\text{-a.s.} \quad \text{for all } \ x \in E. \tag{3.7}$$

In particular $\mathbb{P}_x(\Omega_0) = 1$ for any $x \in E$.

Proof. Let $x \in E$, $n \geq 1$. Then the processes

$$\Big(e^{-s} R_1\big([(1-L)u_n]^+\big)(X_s^0), \mathcal{F}_s^0, \mathbb{P}_x\Big) \quad \text{and} \quad \Big(e^{-s} R_1\big([(1-L)u_n]^-\big)(X_s^0), \mathcal{F}_s^0, \mathbb{P}_x\Big)$$

are positive supermartingales. Indeed since $R_1\big([(1-L)u_n]^\pm\big)$ is continuous by **(H2)**$'$ (ii), the processes are adapted and integrable. The supermartingale property follows by standard manipulations using the simple Markov property. Then by [22, 1.4 Theorem 1] for any $t \geq 0$

$$\exists \lim_{\substack{s \downarrow t \\ s \in S}} e^{-s} \ R_1\big([(1-L)u_n]^\pm\big)(X_s^0) \quad \mathbb{P}_x\text{-a.s.}$$

thus

$$\exists \lim_{\substack{s \downarrow 0 \\ s \in S}} u_n(X_s^0) \quad \mathbb{P}_x\text{-a.s.} \tag{3.8}$$

We have $u_n = R_1\big((1-L)u_n\big)$ and $u_n^2 = R_1\big((1-L_1)u_n^2\big)$ m-a.e., but since both sides are respectively continuous by **(H2)**$'$ (ii), it follows that the equalities hold pointwise on E. Therefore

$$\mathbb{E}_x\big[\big(u_n(X_s^0) - u_n(x)\big)^2\big] = P_s R_1\big((1-L_1)u_n^2\big)(x) - 2u_n(x) P_s R_1\big((1-L)u_n\big)(x) + u_n^2(x)$$

and so

$$\lim_{\substack{s\downarrow 0\\ s\in S}} \mathbb{E}_x\big[\big(u_n(X_s^0)-u_n(x)\big)^2\big]=0 \tag{3.9}$$

by Lemma 3.1.6. (3.8) and (3.9) now imply that

$$\lim_{\substack{s\downarrow 0\\ s\in S}} u_n(X_s^0(\omega))=u_n(x) \quad \text{for all } \omega\in\Omega_x^n, \tag{3.10}$$

where $\Omega_x^n\subset\Omega_1$ with $\mathbb{P}_x(\Omega_x^n)=1$. Let $\omega\in\Omega_x^0:=\bigcap_{n\geq 1}\Omega_x^n$. Then $\mathbb{P}_x(\Omega_x^0)=1$. Suppose that $X_s^0(\omega)$ does not converge to x as $s\downarrow 0$, $s\in S$. Then there exists $\varepsilon_0\in\mathbb{Q}$ and a subsequence $(X_{s_k}^0(\omega))_{k\in\mathbb{N}}$ such that $d(X_{s_k}^0(\omega),x)>\varepsilon_0$ for all $k\in\mathbb{N}$. For $\varepsilon_0\in\mathbb{Q}$ we can find $y\in D$ and u_n in **(H2)$'$** (i) such that $d(x,y)\leq\frac{\varepsilon_0}{4}$ and $u_n(z)\geq 1$, $z\in\overline{B}_{\frac{\varepsilon_0}{4}}(y)$ and $u_n(z)=0$, $z\in E\setminus B_{\frac{\varepsilon_0}{2}}(y)$. Then $u_n(X_{s_k}^0(\omega))$ can not converge to $u_n(x)$ as $k\to\infty$. This is a contradiction. $\square$

Remark 3.1.9. *(i) Let $E=\mathbb{R}^1$ and $(w_n)_{n\in\mathbb{N}}$ be a sequence of functions that separates the points of $\mathbb{R}^1$, i.e. for any $x,y\in\mathbb{R}^1$, $x\neq y$, there exists w_n such that $w_n(x)\neq w_n(y)$. Suppose* (3.10) *holds for $(u_n)_{n\in\mathbb{N}}$ replaced by $(w_n)_{n\in\mathbb{N}}$. Then, we can not conclude* (3.7)*. For instance let*

$$w_n(z)=\begin{cases} z, & \text{if } z\in[-n,n)\\ 2n-z, & \text{if } z\in[n,2n)\\ -2n-z, & \text{if } z\in[-2n,-n)\\ 0, & \text{else.}\end{cases}$$

Then $(w_n)_{n\in\mathbb{N}}$ separates the points of $\mathbb{R}^1$. Let $x=0$ and $X_s^0=\frac{1}{s}$, $\mathbb{P}_x$-a.s. Then for any $n\in\mathbb{N}$, $w_n(X_s^0)$ converges to $w_n(x)$ as $s\downarrow 0$, $\mathbb{P}_x$-a.s. But $X_s^0=\frac{1}{s}$ does not converge to $x=0$ as $s\downarrow 0$.

*(ii) In view of (i) there seems to appear an inconsistency in [12, proof of Lemma 3.8], see also [12, Condition 1.3 (ii) (b)]. Actually the same inconsistency seems to appear in [6, proof of Lemma 4.6]. However, the proof of Lemma 4.6 in [6] can obviously be repaired using a condition similar to our **(H2)$'$** (i) since $C_0^\infty(\mathbb{R}^d)\subset D(L)$ holds in [6].*

Now we define for $t \geq 0$

$$X_t(\omega) := \begin{cases} \lim_{\substack{s \downarrow t \\ s \in S}} X_s^0(\omega) & \text{if } \omega \in \Omega_0 \\ x_0 \quad \text{if } \omega \in \Omega \setminus \Omega_0, \end{cases}$$

where x_0 is an arbitrary but fixed point in E. Then by **(H2)$'$** (iv) for any $t \geq 0$, $f \in C_0(E)$ and $x \in E$

$$\mathbb{E}_x[f(X_t)] = P_t f(x).$$

Since $\sigma(C_0(E)) = \mathcal{B}(E)$, it follows that

$$\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in E_\Delta}),$$

where $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration, is a normal Markov process with transition function $(P_t)_{t \geq 0}$. Moreover, $\mathbb{M}$ has continuous paths up to infinity on E_Δ. The strong Markov property of $\mathbb{M}$ follows from [15, Section I. Theorem (8.11)] using **(H2)$'$** (iii). Hence $\mathbb{M}$ is a Hunt process, i.e. a strong Markov process with continuous sample paths on E_Δ, and has $(P_t)_{t \geq 0}$ as transition function. Therefore **(H2)** holds. Making a statement out of the last conclusion we put it in the following lemma.

Lemma 3.1.10. *Assume* ***(H1)*** *holds. Then* ***(H2)$'$*** *implies* ***(H2)****.*

Remark 3.1.11. *If $(T_t)_{t \geq 0}$ is strong Feller, i.e. $T_t f$ has a continuous m-version for any $f \in \mathcal{B}_b(E)$ and* ***(H2)$'$*** *(i)-(ii) and* ***(H2)$'$*** *(iv) hold, then (**H1**) and (**H2**) hold (cf. Proof of Proposition 3.2.3 below).*

3.1.2 Local setting and general auxiliary results

We assume (**H1**) and (**H2**) throughout the Section 3.1.2.

Definition 3.1.12. *Let B be an open set in E. For $x \in B, t \geq 0, \alpha > 0$ and $p \in [1, \infty)$ let*

- $\sigma_B := \inf\{t > 0 \,|\, X_t \in B\}$,
- $D_B := \inf\{t \geq 0 \,|\, X_t \in B\}$,
- $P_t^B f(x) := \mathbb{E}_x[f(X_t); t < \sigma_{B^c}]$, $f \in \mathcal{B}_b(B)$,
- $R_\alpha^B f(x) := \mathbb{E}_x\Big[\int_0^{\sigma_{B^c}} e^{-\alpha s} f(X_s)\, ds\Big]$, $f \in \mathcal{B}_b(B)$,

- $D(\mathcal{E}^B) := \{u \in D(\mathcal{E}) \mid u = 0 \ \mathcal{E}\text{-}q.e \ on \ B^c\}$.
- $\mathcal{E}^B := \mathcal{E}\,|_{D(\mathcal{E}^B)\times D(\mathcal{E}^B)}$.
- $L^2(B\,, m) := \{u \in L^2(\mathbb{R}^d, m) \mid u = 0, \ m\text{-}a.e. \ on \ B^c\}$.
- $||f||_{p,B}^p := \int_B |f|^p \, dm$.
- $||f||_{\infty,B} := \inf\Big\{c > 0 \mid \int_B 1_{\{|f|>c\}} \, dm = 0\Big\}$.
- $\mathcal{E}_1^B(f,g) := \mathcal{E}^B(f,g) + \int_B fg \, dm, \ \ f,g \in D(\mathcal{E}^B)$.
- $\| f \|_{D(\mathcal{E}^B)} := \mathcal{E}_1^B(f,f)^{1/2}, \ \ f \in D(\mathcal{E}^B)$.

$(\mathcal{E}^B, D(\mathcal{E}^B))$ is called the part Dirichlet form of $(\mathcal{E}, D(\mathcal{E}))$ on B. It is a regular Dirichlet form on $L^2(B,m)$ (cf. [35, Section 4.4]). Let $(T_t^B)_{t>0}$ and $(G_\alpha^B)_{\alpha>0}$ be the $L^2(B,m)$-semigroup and resolvent associated to $(\mathcal{E}^B, D(\mathcal{E}^B))$. Then $P_t^B f, \ R_\alpha^B f$ is an m-version of $T_t^B f, G_\alpha^B f$, respectively for any $f \in L^2(B,m)_b$. Since $P_t^B 1_A(x) \le P_t 1_A(x)$ for any $A \in \mathcal{B}(B)$, $x \in B$ and m has full support on E, $A \mapsto P_t^B 1_A(x), \ A \in \mathcal{B}(B)$ is absolutely continuous with respect to $1_B \cdot m$. Hence there exists a (measurable) transition kernel density $p_t^B(x,y)$, $x, y \in B$, such that

$$P_t^B f(x) = \int_B p_t^B(x,y)\, f(y)\, m(dy), \ t > 0\ , \ \ x \in B \tag{3.11}$$

for $f \in \mathcal{B}_b(B)$. Correspondingly, there exists a (measurable) resolvent kernel density $r_\alpha^B(x,y)$, such that

$$R_\alpha^B f(x) = \int_B r_\alpha^B(x,y)\, f(y)\, m(dy)\,, \ \ \alpha > 0, \ \ x \in B$$

for $f \in \mathcal{B}_b(B)$. For a signed Radon measure μ on B, let us define

$$R_\alpha^B \mu(x) = \int_B r_\alpha^B(x,y)\, \mu(dy)\,, \ \ \alpha > 0, \ \ x \in B$$

whenever this makes sense. The process defined by

$$X_t^B(\omega) = \begin{cases} X_t(\omega), & 0 \le t < D_{B^c}(\omega) \\ \Delta, \quad t \ge D_{B^c}(\omega) \end{cases} \tag{3.12}$$

is called the part process corresponding to $\mathcal{E}^B$ and is denoted by $\mathbb{M}|_B$. $\mathbb{M}|_B$ is a Hunt process on B (see [35, p.174 and Theorem A.2.10]). In particular, by (3.11) $\mathbb{M}|_B$ satisfies the *absolute continuity condition* on B.
A positive Radon measure μ on B is said to be of finite energy integral if

$$\int_B |f(x)|\,\mu(dx) \le C\sqrt{\mathcal{E}_1^B(f,f)},\ f \in D(\mathcal{E}^B)\cap C_0(B),$$

where C is some constant independent of f. A positive Radon measure μ on B is of finite energy integral (on B) if and only if there exists a unique function $U_1^B\,\mu \in D(\mathcal{E}^B)$ such that

$$\mathcal{E}_1^B(U_1^B\,\mu, f) = \int_B f(x)\,\mu(dx),$$

for all $f \in D(\mathcal{E}^B)\cap C_0(B)$. $U_1^B\,\mu$ is called 1-potential of μ. In particular, $R_1^B\mu$ is a version of $U_1^B\mu$ (see e.g. [35, Exercise 4.2.2]). The measures of finite energy integral are denoted by S_0^B. We further define $S_{00}^B := \{\mu \in S_0^B \mid \mu(B) < \infty, \|U_1^B\mu\|_{\infty,B} < \infty\}$. A positive Borel measure μ on B is said to be smooth in the strict sense if there exists a sequence $(E_k)_{k\ge1}$ of Borel sets increasing to B such that $1_{E_k}\cdot\mu \in S_{00}^B$ for each k and

$$\mathbb{P}_x(\lim_{k\to\infty}\sigma_{B\setminus E_k} \ge \zeta) = 1\ ,\ \ \forall x \in B.$$

The totality of the smooth measures in the strict sense is denoted by S_1^B (see [35]). If $\mu \in S_1^B$, then there exists a unique $A \in A_{c,1}^{+,B}$ with $\mu = \mu_A$, i.e. μ is the Revuz measure of A (see [35, Theorem 5.1.7]), such that

$$\mathbb{E}_x\Big[\int_0^\infty e^{-t}\,dA_t\Big] = R_1\mu_A(x)\ ,\ \ \forall x \in B.$$

Here, $A_{c,1}^{+,B}$ denotes the positive continuous additive functionals on B in the strict sense. If $B = E$, we omit the superscript B and simply write U_1, S_0, S_{00}, S_1, and $A_{c,1}^{+}$.

Lemma 3.1.13. *For $k \in \mathbb{Z}$, let $\mu_{A^k}, \mu_A \in S_1^B$ be the Revuz measures associated with $A^k, A \in A_{c,1}^{+,B}$, respectively. Suppose that $\mu_A = \sum_{k\in\mathbb{Z}}\mu_{A^k}$. Then $A = \sum_{k\in\mathbb{Z}} A^k$.*

Proof. Since $\mu_{\sum_{-n\le k\le n}A^k} \le \mu_A$ and $\sum_{-n\le k\le n} A^k \in S_1^B$, we can use [15, IV. (2.12) Proposition] in order to show that for any $n \in \mathbb{N}$ and $t \ge 0$

$$\mathbb{P}_x\Big(\sum_{-n\le k\le n} A_t^k \le A_t\Big) = 1$$

for all $x \in B$. Thus by the Weierstrass M-test $\tilde{N}_t := \sum_{k\in\mathbb{Z}} A_t^k$ converges locally uniformly $\mathbb{P}_x$-a.s. for all $x \in B$. It follows that $\tilde{N}_t$ is positive continuous additive functional in the strict sense. In particular $d\tilde{N}_t = \sum_{k\in\mathbb{Z}} dA_t^k$ which further implies that for any $x \in B$ and $f \in C_0(B)$

$$\begin{aligned} &\mathbb{E}_x\Big[\int_0^\infty e^{-t} f(X_t)\, d\tilde{N}_t\Big] = \sum_{k\in\mathbb{Z}} \mathbb{E}_x\Big[\int_0^\infty e^{-t} f(X_t)\, dA_t^k\Big] = \sum_{k\in\mathbb{Z}} R_1^B f\mu_{A^k}(x) \\ = &\sum_{k\in\mathbb{Z}} \int_E r_1^B(x,y)\, f(y)\ \mu_{A^k}(dy) = \int_E r_1^B(x,y)\, f(y)\ \mu_A(dy) = R_1^B f\mu_A(x) \\ = &\ \mathbb{E}_x\Big[\int_0^\infty e^{-t} f(X_t)\, dA_t\Big]. \end{aligned}$$

Hence, $\tilde{N} = A$ by [15, IV. (2.12) Proposition]. □

Proposition 3.1.14. *Let μ be a positive Radon measure on E. Suppose that for some relatively compact open set $G \subset E$, $1_G \cdot \mu \in S_0$ and that $R_1(1_G \cdot \mu)$ is bounded m-a.e. on E by a continuous function $r_1^G \in C(E)$ (resp. that $R_1(1_G \cdot \mu) \in L^1(G, \mu)$ and that $R_1(1_G \cdot \mu)$ is bounded m-a.e. on E by a continuous function $r_1^G \in C(E)$). Then $1_G \cdot \mu \in S_{00}$. In particular, if this holds for any relatively compact open set G, then $\mu \in S_1$ with respect to a sequence of open sets $(E_k)_{k\geq 1}$.*

Proof. First suppose $1_G \cdot \mu \in S_0$. Since μ is a Radon measure, we have that $1_G \cdot \mu$ is finite. Since r_1^G is continuous, it follows that

$$E_k := \{x \in E \mid r_1^G(x) < k\}, \quad k \geq 1$$

are open sets increasing to E. Let $\tilde{U}_1(1_{E_k\cap G}\cdot\mu)$, $\tilde{U}_1(1_G\cdot\mu)$ be q.c. versions of $U_1(1_{E_k\cap G}\cdot \mu)$, $U_1(1_G \cdot \mu)$. On E_k it holds $\tilde{U}_1(1_{E_k\cap G} \cdot \mu) \leq \tilde{U}_1(1_G \cdot \mu) \leq r_1^G \leq k$ q.e. Hence $U_1(1_{E_k\cap G} \cdot \mu) \leq k$ m-a.e. by [35, Lemma 2.2.4 (ii)]. Since $(E_k)_{k\geq 1}$ is an open cover of $\overline{G}$, we know that there exists $k_0 \in \mathbb{N}$ with $G \subset \overline{G} \subset E_{k_0}$. Hence, $U_1\,(1_G \cdot \mu) \leq k_0$ m-a.e. Therefore, $1_G \cdot \mu \in S_{00}$. If $R_1(1_G \cdot \mu) \in L^1(G, \mu)$, then

$$\int_G \int_G r_1(x,y)\ \mu(dy)\ \mu(dx) = \int_G R_1(1_G \cdot \mu)(x)\ \mu(dx) < \infty.$$

Hence $1_G \cdot \mu \in S_0$ by [35, Example 4.2.2] and we conclude as before. □

3.2 Muckenhoupt A_2-weights

In this section we complete and extend substantially the results from Chapter 2. We assume throughout that $E = \mathbb{R}^d$, with $d \geq 3$ (except in Lemma 3.2.6(vi), Proposition 3.2.8(ii), Theorem 3.2.9(ii) and Remark 3.2.16 where the state space is $\mathbb{R}^d \setminus \{0\}$). We consider a weight function that is in the Muckenhoupt A_2 class. For the definition and basic properties of Muckenhoupt weights, we refer to [72]. Precisely, we assume the following:

(α) $\phi : \mathbb{R}^d \to [0, \infty)$ is a $\mathcal{B}(\mathbb{R}^d)$-measurable function and $\phi > 0$ dx-a.e.,

(β) $\rho\phi \in A_2$, $\rho \in H^{1,1}_{loc}(\mathbb{R}^d, dx)$, $\rho > 0$ dx-a.e.,

and consider

$$\mathcal{E}(f,g) := \frac{1}{2} \int_{\mathbb{R}^d} \nabla f \cdot \nabla g \; dm, \quad f, g \in C_0^\infty(\mathbb{R}^d), \quad m := \rho\phi dx \tag{3.13}$$

in $L^2(\mathbb{R}^d, m)$.

Remark 3.2.1. *Let $\tilde{c} \geq 1$. If ϕ is measurable with $\tilde{c}^{-1} \leq \phi \leq \tilde{c}$ and $\rho \in A_2$, then $\rho\phi \in A_2$.*

Since $\rho\phi \in A_2$, we have $\frac{1}{\rho\phi} \in L^1_{loc}(\mathbb{R}^d, dx)$, and the latter implies that (3.13) is closable in $L^2(\mathbb{R}^d, m)$ (see [44, II.2 a)]). The closure $(\mathcal{E}, D(\mathcal{E}))$ of (3.13) is a strongly local, regular, symmetric Dirichlet form (cf. e.g. [63, p. 274]).
From [62, p. 303 Proposition 2.3] and [63, p. 286 A)] (see also [63, 5.B] and [14]) we know that there exists a jointly continuous transition kernel density $p_t(x, y)$ such that

$$P_t f(x) := \int_{\mathbb{R}^d} p_t(x, y)\, f(y)\; m(dy), \quad t > 0, \; x, y \in \mathbb{R}^d, \; f \in \mathcal{B}_b(\mathbb{R}^d)$$

is an m-version of $T_t f$ if $f \in L^2(\mathbb{R}^d, m)_b$. We want to show that $(P_t)_{t \geq 0}$ is strong Feller. For this, we first need a lemma.

Lemma 3.2.2. *Let $t, r > 0$. Then $\inf_{x \in \bar{B}_r} m\big(B_{\sqrt{t}}(x)\big) =: M_{t,r} > 0$ and for any $x \in \bar{B}_r$,*

$\varepsilon > 0$

$$p_t(x,y) \le \frac{c\exp\Big(-\frac{\|y\|^2}{2(4+\varepsilon)t}\Big)\ \Big(1+\frac{\|y\|}{\sqrt{t}}\Big)^{\alpha/2}}{M_{t,r}^{1/2}\ m\big(B_{\sqrt{t}}(0)\big)^{1/2}}\ 1_{\mathbb{R}^d\setminus\bar{B}_{4r}}(y) + \Big(\sup_{\substack{x\in\bar{B}_r\\ y\in\bar{B}_{4r}}} p_t(x,y)\Big)1_{\bar{B}_{4r}}(y) \quad (3.14)$$

where $\alpha > 0$ is some constant. In particular

$$\sup_{x\in\bar{B}_r} p_t(x,\cdot) \in L^1(\mathbb{R}^d, m).$$

Proof. It follows from [64, 4.3] and [63, Corollary 4.2.] that for $x, y \in \mathbb{R}^d$, $t > 0$ and any $\varepsilon > 0$

$$p_t(x,y) \le c\ \frac{\exp\Big(-\frac{\|x-y\|^2}{(4+\varepsilon)t}\Big)}{m\big(B_{\sqrt{t}}(x)\big)^{1/2} m\big(B_{\sqrt{t}}(y)\big)^{1/2}}. \qquad (3.15)$$

By Fatou's lemma, $x \mapsto m\big(B_{\sqrt{t}}(x)\big)$ is lower semicontinuous and so it attains its infimum on $\bar{B}_r$. Therefore $M_{t,r} > 0$. Moreover, since $-\|x-y\|^2 \le -\frac{\|y\|^2}{2} + \frac{\|y\|(4\|x\|-\|y\|)}{2}$, we obtain $-\|x-y\|^2 \le -\frac{\|y\|^2}{2}$ for any $x \in \bar{B}_r$ if $y \in \mathbb{R}^d \setminus \bar{B}_{4r}$. Further for some $\alpha > 0$ and any $x, y \in \mathbb{R}^d$

$$m\big(B_{\sqrt{t}}(y)\big) \ge \frac{m(B_{\sqrt{t}}(x))}{C_D}\left(1+\frac{\|x-y\|}{\sqrt{t}}\right)^{-\alpha}, \qquad (3.16)$$

where C_D is the volume doubling constant of m (see [38, Proposition 5.1]). These facts together with the joint continuity of $p_t(x,y)$ and (3.15) lead to (3.14). Since $m\big(B_r(y)\big)$ has at most polynomial growth in r, $r \ge 1$ for any $y \in \mathbb{R}^d$ (cf. proof of Proposition 2.1.4) the last statement follows. □

Proposition 3.2.3. *(i) $(P_t)_{t\ge0}$ (resp. $(R_\alpha)_{\alpha>0}$) is strong Feller, i.e. for $t > 0$, we have $P_t(\mathcal{B}_b(\mathbb{R}^d)) \subset C_b(\mathbb{R}^d)$ (resp. for $\alpha > 0$, we have $R_\alpha(\mathcal{B}_b(\mathbb{R}^d)) \subset C_b(\mathbb{R}^d)$).*

*(ii) (**H1**) and **(H2)′** (iii) and (iv) hold for $(P_t)_{t\ge0}$.*

(iii) $P_t(L^1(\mathbb{R}^d, m)_0) \subset C_\infty(\mathbb{R}^d)$.

(iv) Let μ be a positive Radon measure and $G \subset \mathbb{R}^d$ relatively compact open. Let

$$\int_G r_1(\cdot, y)\ \mu(dy) \le r_1^G$$

μ-a.e. on G and m-a.e. on $\mathbb{R}^d$, where r_1^G is a continuous function on $\mathbb{R}^d$. Then $1_G \cdot \mu \in S_{00}$.

Proof. (i) Let $x_n \to x$ in $\mathbb{R}^d$. For $f \in \mathcal{B}_b(\mathbb{R}^d)$ and $t > 0$

$$|P_t f(x_n) - P_t f(x)| \leq \int_{\mathbb{R}^d} |p_t(x_n, y) - p_t(x, y)| \; |f(y)| \; m(dy)$$

which converges to 0 by Lebesgue in view of Lemma 3.2.2 and the continuity of $p_t(\cdot, y)$. Clearly, $P_t f$ is bounded. Hence, $(P_t)_{t \geq 0}$ is strong Feller. Since $R_\alpha f(x) = \int_0^\infty e^{-t} \; P_t f(x) \; dt$ and $\|P_t f\|_\infty \leq \|f\|_\infty$ for any $f \in \mathcal{B}_b(\mathbb{R}^d)$, $(R_\alpha)_{\alpha > 0}$ is clearly also strong Feller by Lebesgue.
(ii) By (i), $A \mapsto P_t(x, A)$ is a sub-probability measure on $\mathcal{B}(\mathbb{R}^d)$ for any $t > 0$, $x \in \mathbb{R}^d$. Obviously, $x \mapsto P_t(x, A)$ is also measurable for any $A \in \mathcal{B}(\mathbb{R}^d)$ and so it remains to show the Chapman-Kolmogorov equation. By the semigroup property,

$$P_{t+s} 1_A(x) = P_t(P_s 1_A)(x), \quad A \in \mathcal{B}(\mathbb{R}^d), \; t, s > 0 \tag{3.17}$$

for m-a.e.$x \in \mathbb{R}^d$. From the strong Feller property, both sides of (3.17) are continuous, hence (3.17) holds for every $x \in \mathbb{R}^d$, i.e. the Chapman-Kolmogorov equation holds and so $(P_t)_{t \geq 0}$ is a sub-Markovian transition function. **(H2)**$'$ (iii) follows from (i) and **(H2)**$'$ (iv) follows from [63, Proposition 3.1].
(iii) Combining (3.15) and (3.16) we have for any $x, y \in \mathbb{R}^d$, $t > 0$ and $\varepsilon > 0$,

$$p_t(x, y) \leq c \frac{1}{m\Big(B_{\sqrt{t}}(y)\Big)} \; \exp\Big(- \frac{\|x - y\|^2}{(4 + \varepsilon)t} \Big). \tag{3.18}$$

Using the joint continuity of $p_t(\cdot, \cdot)$, as in (i) we can see that $P_t(L^1(\mathbb{R}^d, m)_0) \subset C(\mathbb{R}^d)$. Let $f \in L^1(\mathbb{R}^d, m)_0$. Using (3.18),

$$\begin{aligned} \big|P_t f(x)\big| &\leq \int_{\mathbb{R}^d} |f(y)| \; p_t(x, y) \; m(dy) \\ &\leq \frac{c}{\inf_{y \in \mathrm{supp}(f)} m(B_{\sqrt{t}}(y))} \int_{\mathrm{supp}(f)} |f(y)| \; e^{-\frac{\|x-y\|^2}{(4+\varepsilon)t}} \; m(dy) \end{aligned}$$

which converges to 0 by Lebesgue as $x \to \infty$.
(iv) This is just a reformulation of Proposition 3.1.14. □

First let us assume that

(γ) The transition function $(P_t)_{t\geq 0}$ satisfies (**H2**) with $E=\mathbb{R}^d$.

Later we will use the Feller semigroup method and the Dirichlet form method for some typical Muckenhoupt A_2 weights to verify (γ). By the existence of $\mathbb{M}$ associated with $(P_t)_{t\geq 0}$, $\mathbb{M}$ satisfies the *absolute continuity condition*. Since $\rho\phi \in A_2$, $(\mathcal{E}, D(\mathcal{E}))$ is *conservative*, i.e. $T_t 1(x) = 1$ for m-a.e. $x \in \mathbb{R}^d$ and all $t > 0$ (see Proposition 2.1.4). It follows

$$\mathbb{P}_x(\zeta = \infty) = 1, \quad \forall x \in \mathbb{R}^d, \tag{3.19}$$

by [35, Theorem 4.5.4 (iv)] and

$$\mathbb{P}_x\big(t \mapsto X_t \text{ is continuous on } [0,\infty)\big) = 1, \quad \forall x \in \mathbb{R}^d, \tag{3.20}$$

by [35, Theorem 4.5.4 (ii)].
Throughout, let $f^j(x) := x_j$, $j = 1, \dots, d$, $x \in \mathbb{R}^d$, be the coordinate projections. In order to be explicit, we further assume the following integrations by parts formula

(IBP) For $f \in \{f^1, \dots, f^d\}$, $g \in C_0^\infty(\mathbb{R}^d)$

$$-\mathcal{E}(f,g) = \int_{\mathbb{R}^d} \left(\nabla f \cdot \frac{\nabla \rho}{2\rho}\right) g \; dm + \int_{\mathbb{R}^d} g \, d\nu^f,$$

where $\nu^f = \sum_{k\in\mathbb{Z}} \nu_k^f$ and $\nu^f, \nu_k^f, k \in \mathbb{Z}$ are signed Radon measures (locally of bounded total variation).

For a signed Radon measure μ we denote by μ^+ and μ^- the positive and negative parts in the Hahn decomposition for μ, i.e. $\mu = \mu^+ - \mu^-$. Additionally, we assume that

(δ) For any $G \subset \mathbb{R}^d$ relatively compact open, $k \in \mathbb{Z}$ and $f \in \{f^1, \dots, f^d\}$, we have that $1_G \cdot \nu^{f+}$, $1_G \cdot \nu^{f-}$, $1_G \cdot \nu_k^{f+}$, $1_G \cdot \nu_k^{f-}, 1_G \cdot \frac{\|\nabla\rho\|}{\rho} m \in S_0$ and the corresponding 1-potentials are all bounded by continuous functions.

Theorem 3.2.4. *Suppose* $(\alpha) - (\delta)$ *and (IBP). Then*

$$X_t = x + W_t + \int_0^t \frac{\nabla \rho}{2\,\rho}(X_s)\, ds + \sum_{k\in\mathbb{Z}} L_t^k\,, \quad t \geq 0, \tag{3.21}$$

$\mathbb{P}_x$-a.s. for any $x \in \mathbb{R}^d$ where W is a standard d-dimensional Brownian motion starting from zero, $L^k = (L^{1,k}, \dots, L^{d,k})$ and $L^{j,k}, j = 1, \dots, d$, is the difference of positive continuous additive functionals of X in the strict sense associated with Revuz measure $\nu_k^{f^j} = \nu_k^{f^j,(1)} - \nu_k^{f^j,(2)}$ defined in (IBP) (cf. [35, Theorem 5.1.3]).

Proof. Given that $(\alpha) - (\delta)$ and (IBP) hold, the assertion follows from [35, Theorems 5.1.3 and 5.5.5], Lemma 3.1.13, and Propositions 3.2.3 and 3.1.14. $\square$

For later purpose we add some auxiliary results. Define

$$V_\eta g(x) := \int_{\mathbb{R}^d} \frac{1}{\|x-y\|^{d-\eta}} \, g(y) \, dy, \quad x \in \mathbb{R}^d, \ \eta > 0, \tag{3.22}$$

whenever it makes sense.

Lemma 3.2.5. *Let $\eta \in (0, d)$, $0 < \eta - \frac{d}{p} < 1$ and $g \in L^p(\mathbb{R}^d, dx)$ with*

$$\int_{\mathbb{R}^d} (1 + \|y\|)^{\eta - d} |g(y)| \, dy < \infty.$$

Then $V_\eta g$ is Hölder continuous of order $\eta - \frac{d}{p}$.

Proof. See [45, Chapter 4, Theorem 2.2]. $\square$

Lemma 3.2.6. *Let $\tilde{c}^{-1}\|x\|^\alpha \le \rho\phi(x) \le \tilde{c}\|x\|^\alpha$ for some $\alpha \in (-d, d)$, $\tilde{c} \ge 1$. Then:*

*(i) $\lim_{t \downarrow 0} P_t f(x) = f(x)$, $\forall x \in \mathbb{R}^d$, $\forall f \in C_0(\mathbb{R}^d)$, i.e. (**H1**) and (**H2**) hold (cf. Proposition 3.2.3(i),(iii) and Lemma 3.1.3).*

(ii) Let $\Phi(x, y) := \frac{1}{\|x-y\|^{\alpha+d-2}}$ and $\Psi(x, y) := \frac{1}{\|x-y\|^{d-2}\|y\|^\alpha}$. Then

$$c^{-1} \left(\Phi(x, y) + \Psi(x, y) 1_{\{\alpha \in [0,d)\}} \right) \le r_1(x, y) \le c \left(\Phi(x, y) + \Psi(x, y) 1_{\{\alpha \in (-d,0)\}} \right).$$

(iii) Let $\alpha \in (-d+1, 2)$ and $G \subset \mathbb{R}^d$ any relatively compact open set. Suppose $1_G \cdot f \, \|x\|^\alpha \in L^p(\mathbb{R}^d, dx)$, $p \ge 1$ with $0 < 2 - \alpha - \frac{d}{p} < 1$ and $1_G \cdot f \in L^q(\mathbb{R}^d, dx)$ with $0 < 2 - \frac{d}{q} < 1$. Then $R_1(1_G \cdot |f| m)$ is bounded everywhere (hence clearly also bounded m-a.e. on $\mathbb{R}^d$ and $R_1(1_G \cdot |f| m) \in L^1(G, |f| m)$) by the continuous function $\int_G |f(y)| \ (\Phi(\cdot, y) + \Psi(\cdot, y)) \ m(dy)$. In particular, Proposition 3.1.14 applies and $1_G \cdot |f| m \in S_{00}$.

(iv) *Let* $\alpha \in (-d+1,2)$. *Then* $R_1\left(1_G \cdot \frac{\|\nabla\rho\|}{\rho} m\right)$ *is pointwise bounded by a continuous function for any relatively compact open set* $G \subset \mathbb{R}^d$. *In particular* $1_G \cdot \frac{\|\nabla\rho\|}{\rho} m \in S_{00}$ *for any relatively compact open set* $G \subset \mathbb{R}^d$.

(v) *Let* $\alpha \in (-d+1,1)$. *Let* $D \subset \mathbb{R}^d$ *be a bounded Lipschitz domain with surface measure* $\sigma_{\partial D}$. *Suppose that* ρ *is bounded on* ∂D *(more precisely the trace of* ρ *on* ∂D*, which exists since* $\rho \in H^{1,1}_{loc}(\mathbb{R}^d)$*). Then* $R_1(1_G \cdot \rho\sigma_{\partial D})$ *is pointwise bounded by a continuous function for any relatively compact open set* $G \subset \mathbb{R}^d$. *In particular* $1_G \cdot \rho\sigma_{\partial D} \in S_{00}$ *for any relatively compact open* $G \subset \mathbb{R}^d$.

(vi) *Let* $\alpha \in [-d+2,d)$. *Then* $\mathrm{Cap}(\{0\}) = 0$ *and the part Dirichlet form* $(\mathcal{E}^B, D(\mathcal{E}^B))$ *on* $B := \mathbb{R}^d \setminus \{0\}$ *satisfies (**H1**), (**H2**) with transition kernel density* $p_t^B = p_t|_{B\times B}$. *Moreover* $(\mathcal{E}^B, D(\mathcal{E}^B))$ *is conservative.*

Proof. (i) From Proposition 3.2.3, we know that $(P_t)_{t\geq 0}$ satisfies (**H1**) and is strong Feller. Thus Lemma 3.1.3 (ii) holds. We will check Lemma 3.1.3 (i). Let $m_\alpha := \|y\|^\alpha dy$, $\alpha \in (-d,d)$. For $\alpha \in [0,d)$ and $0 < \sqrt{t} \leq \|x\|$, we have

$$\begin{aligned} & m(B_{\sqrt{t}}(x)) \geq \tilde{c}^{-1}\ m_\alpha(B_{\sqrt{t}}(x)) \\ \geq\ & \tilde{c}^{-1} \inf_{y\in B_{\sqrt{t}}(x)} \|y\|^\alpha\ vol(B_{\sqrt{t}}(x)) \geq c_d\tilde{c}^{-1}\ (\|x\| - \sqrt{t})^\alpha\ \sqrt{t}^d, \end{aligned} \tag{3.23}$$

with $c_d = vol(B_1(0))$, and for $\alpha \in (-d,0)$ and $0 < \sqrt{t} \leq \|x\|$, we have

$$\begin{aligned} & m(B_{\sqrt{t}}(x)) \geq \tilde{c}^{-1}\ m_\alpha(B_{\sqrt{t}}(x)) \geq \tilde{c}^{-1} \frac{(c_d\sqrt{t}^d)^2}{m_{-\alpha}(B_{\sqrt{t}}(x))} \\ \geq\ & \tilde{c}^{-1} \frac{(c_d\sqrt{t}^d)^2}{c_d\ \sqrt{t}^d\ (\|x\| + \|\sqrt{t}\|)^{-\alpha}} \geq \tilde{c}^{-1} c_d\ \sqrt{t}^d\ (2\|x\|)^\alpha. \end{aligned} \tag{3.24}$$

Since $(\mathcal{E}, D(\mathcal{E}))$ is conservative and $(P_t)_{t\geq 0}$ is strong Feller, we have $P_t 1(x) = 1$ for all

$x \in \mathbb{R}^d$, $t > 0$. Thus by (3.23), symmetry of $p_t(x, y)$ in (x, y), and (3.18), we get

$$\begin{aligned} \big|P_t f(x) - f(x)\big| &= \Big| \int_{\mathbb{R}^d} \big(f(y) - f(x)\big)\, p_t(x, y)\ m(dy)\Big| \\ &\leq\ c\int_{\mathbb{R}^d} \big|f(x + \sqrt{t}y) - f(x)\big| \frac{\exp\left(-\frac{\|y\|^2}{4+\varepsilon}\right)\ (\sqrt{t})^d}{m\left(B_{\sqrt{t}}(x)\right)} \|x + \sqrt{t}y\|^\alpha\ dy \\ &\leq\ c\int_{\mathbb{R}^d} \big|f(x + \sqrt{t}y) - f(x)\big| \frac{\exp\left(-\frac{\|y\|^2}{4+\varepsilon}\right)\ (\sqrt{t}^d)}{(\|x\| - \sqrt{t})^\alpha \sqrt{t}^d}\ \|x + \sqrt{t}y\|^\alpha\ dy, \end{aligned}$$

which converges to 0 as $t \to 0$. For $x = 0$, by (3.18) and symmetry of $p_t(\cdot, \cdot)$, we get

$$\begin{aligned} &\big|P_t f(0) - f(0)\big| \\ \leq&\ c\int_{\mathbb{R}^d} \big|f(\sqrt{t}y) - f(0)\big| \frac{\exp\left(-\frac{\|y\|^2}{4+\varepsilon}\right)\ (\sqrt{t})^d}{m\left(B_{\sqrt{t}}(0)\right)} \|\sqrt{t}y\|^\alpha\ dy \\ \leq&\ c\int_{\mathbb{R}^d} \big|f(\sqrt{t}y) - f(0)\big| \frac{\exp\left(-\frac{\|y\|^2}{4+\varepsilon}\right)\ (\sqrt{t})^d}{(\sqrt{t})^{\alpha+d}}\ \|\sqrt{t}y\|^\alpha\ dy, \end{aligned}$$

which also converges to 0 as $t \to 0$. For $\alpha \in (-d, 0)$, using (3.24) instead of (3.23), similarly to the case of $\alpha \in [0, d)$ one can show that $P_t f(x) \to f(x)$ as $t \to 0$. Thus Lemma 3.1.3 (i) holds.
(ii) For $\alpha \in [0, d)$, we have

$$c\ \sqrt{t}^{\alpha+d} \leq m(B_{\sqrt{t}}(x)) \leq c\ \sqrt{t}^d\ (\|x\| + \sqrt{t})^\alpha, \tag{3.25}$$

and for $\alpha \in (-d, 0)$,

$$c\ \sqrt{t}^d\ (\|x\| + \sqrt{t})^\alpha \leq m\left(B_{\sqrt{t}}(x)\right) \leq c\sqrt{t}^{d+\alpha}. \tag{3.26}$$

By [63, Corollary 4.10] and (3.18)

$$\frac{1}{c\ m\left(B_{\sqrt{t}}(y)\right)}\ \exp\Big(-c\frac{\|x-y\|^2}{t}\Big) \leq p_t(x, y) \leq \frac{c}{m\left(B_{\sqrt{t}}(y)\right)}\ \exp\Big(-\frac{\|x-y\|^2}{5t}\Big).$$

Let first $\alpha \in [0,d)$. Then, for $x, y \in \mathbb{R}^d$ using the first inequality in (3.25), we get

$$r_1(x,y) \leq \int_0^\infty \frac{c}{(\sqrt{t})^{\alpha+d}} \exp\Big(-\frac{\|x-y\|^2}{5t}\Big)\, dt.$$

By standard calculations, using a change of variable with $s = \frac{\|x-y\|^2}{t}$, we obtain

$$r_1(x,y) \leq \frac{c}{\|x-y\|^{\alpha+d-2}}. \tag{3.27}$$

Using the second inequality in (3.25), we get the lower bound of $r_1(x,y)$,

$$\begin{aligned} r_1(x,y) \;&\geq\; \int_0^\infty \frac{c}{\sqrt{t}^d\ (\|y\|+\sqrt{t})^\alpha} \exp\Big(-c\frac{\|x-y\|^2}{t}\Big)\, dt \\ &\geq\; \int_0^{\|y\|^2} \frac{c}{\sqrt{t}^d\ (2\|y\|)^\alpha} \exp\Big(-c\frac{\|x-y\|^2}{t}\Big)\, dt \\ &\quad + \int_{\|y\|^2}^\infty \frac{c}{\sqrt{t}^d\ (2\sqrt{t})^\alpha} \exp\Big(-c\frac{\|x-y\|^2}{t}\Big)\, dt. \end{aligned}$$

Hence,

$$r_1(x,y) \geq c\left(\frac{1}{\|x-y\|^{\alpha+d-2}} + \frac{1}{\|x-y\|^{d-2}\|y\|^\alpha}\right).$$

For $\alpha \in (-d, 0)$, using (3.26) instead of (3.25), we get

$$\frac{c}{\|x-y\|^{\alpha+d-2}} \leq r_1(x,y) \leq c\left(\frac{1}{\|x-y\|^{\alpha+d-2}} + \frac{1}{\|x-y\|^{d-2}\|y\|^\alpha}\right).$$

(iii) If $\alpha \in (-d+2, 2)$, then the conditions imply that $V_{2-\alpha}\left(1_G \cdot f\|x\|^\alpha\right)$ as well as $V_2\left(1_G \cdot f\right)$ are continuous by Lemma 3.2.5. If $\alpha \in (-d+1, -d+2]$, then $V_{2-\alpha}\left(1_G \cdot f\|x\|^\alpha\right)$ is easily seen to be also continuous and so by (ii) for any $x \in \mathbb{R}^d$

$$R_1(1_G \cdot |f|m)(x) \leq c\ \big(\ V_{2-\alpha}(1_G \cdot |f|\ \|x\|^\alpha)(x) + V_2(1_G \cdot |f|)(x)\big).$$

(iv) Let $\alpha \in (-d+2, 2)$ and $0 < \varepsilon < 1$ satisfy $2 - \varepsilon > \alpha$. Then $1_G \cdot \frac{\|\nabla\rho\|}{\rho}\|x\|^\alpha = c\, 1_G \cdot \|x\|^{\alpha-1} \in L^p(\mathbb{R}^d, dx)$ for $p = \frac{d}{(2-\varepsilon)-\alpha} \geq 1$ and $1_G \cdot \frac{\|\nabla\rho\|}{\rho} = c\, 1_G \cdot \|x\|^{-1} \in L^q(\mathbb{R}^d, dx)$ for $q = \frac{2d}{3}$. For $\alpha \in (-d+1, -d+2]$, $V_{2-\alpha}\left(1_G \cdot \frac{\|\nabla\rho\|}{\rho}\|x\|^\alpha\right)$ is continuous. Therefore, the statement follows as in (iii).

(v) Let G be relatively compact open. There exist $B_i \subset \partial D$ and Lipschitz continuous functions F_i, $i = 1, \dots n$ for some $n \in \mathbb{N}$ such that $B_i = \{x \in \partial D \mid x = (x', F_i(x')) \in \mathbb{R}^{d-1} \times \mathbb{R}\}$ and $\bigcup_{i=1}^n B_i = \partial D$. Set $B_i^* = \{y' \in \mathbb{R}^{d-1} \mid (y', F_i(y')) \in B_i)\}$. Let first $\alpha \in [0, 1)$. Then, using (ii) we get for every $x \in \mathbb{R}^d$

$$\begin{aligned} R_1(1_G \cdot \rho\sigma_{\partial D})(x) &\leq c \int_{G \cap \partial D} \frac{1}{\|x-y\|^{\alpha+d-2}} \, \sigma_{\partial D}(dy) \\ &\leq c \sum_{i=1}^n \int_{B_i^*} \frac{\sqrt{1 + |\nabla F_i(y')|^2}}{\|x'-y'\|^{(\alpha+d-2)}} \, dy' \\ &\leq c \sum_{i=1}^n \int_{B_i^*} \frac{1}{\|x'-y'\|^{(\alpha+d-2)}} \, dy' \\ &\leq c \int_K \frac{1}{\|x'-y'\|^{\alpha+d-2}} \, dy', \end{aligned}$$

where $K \subset \mathbb{R}^{d-1}$ is some compact set. Since the last expression is continuous in x' (hence in particular in x) by Lemma 3.2.5, the final statement follows by Proposition 3.1.14. For $\alpha \in (-d+1, 0)$ we have for any $x \in \mathbb{R}^d$ that

$$\begin{aligned} R_1(1_G \cdot \rho\sigma_{\partial D})(x) &= \int_{\partial D \cap G} r_1(x,y) \, \rho(y) \, \sigma_{\partial D}(dy) \\ &\leq c \int_{\partial D} \left(\frac{1}{\|y\|^\alpha \|x-y\|^{d-2}} + \frac{1}{\|x-y\|^{\alpha+d-2}} \right) \|y\|^\alpha \, \sigma_{\partial D}(dy) \\ &\leq c \int_{\partial D} \left(\frac{1}{\|x-y\|^{d-2}} + \frac{1}{\|x-y\|^{\alpha+d-2}} \right) \sigma_{\partial D}(dx), \end{aligned}$$

and we conclude as before in the case of $\alpha \in [0, 1)$.
(vi) By [35, Example 3.3.2] it holds $\mathrm{Cap}(\{0\}) = 0$. Hence $u(x) := \mathbb{P}_x(\sigma_{B^c} < \infty) = 0$ for m-a.e. $x \in \mathbb{R}^d$. Since u is an excessive function and $\mathbb{M}$ satisfies the absolute continuity condition it follows $u(x) = 0$ for all $x \in \mathbb{R}^d$. From this the remaining part of the proof is straightforward. □

3.2.1 Skew reflection on spheres and on a Lipschitz domain

Skew reflection on spheres

In Chapter 2, we considered the Dirichlet form determined by (3.13) with concrete ϕ and $\rho = \xi^2$, $\xi \in H^{1,2}_{loc}(\mathbb{R}^d)$. More precisely, our assumptions were the followings: we let

$m_0 \in (0, \infty)$ and $(l_k)_{k\in\mathbb{Z}} \subset (0, m_0)$, $0 < l_k < l_{k+1} < m_0$, be a sequence converging to 0 as $k \to -\infty$ and converging to m_0 as $k \to \infty$, $(r_k)_{k\in\mathbb{Z}} \subset (m_0, \infty)$, $m_0 < r_k < r_{k+1} < \infty$, be a sequence converging to m_0 as $k \to -\infty$ and tending to infinity as $k \to \infty$, and set

$$\phi := \sum_{k\in\mathbb{Z}} \Big(\gamma_k \cdot 1_{A_k} + \overline{\gamma}_k \cdot 1_{\hat{A}_k} \Big), \tag{3.28}$$

where γ_k , $\overline{\gamma}_k \in (0, \infty)$, $A_k := B_{l_k} \setminus \overline{B}_{l_{k-1}}$, $\hat{A}_k := B_{r_k} \setminus \overline{B}_{r_{k-1}}$. We further assumed

(a) $\rho\phi \in A_2$, $\rho = \xi^2$, $\xi \in H^{1,2}_{loc}(\mathbb{R}^d)$, $\rho > 0$ dx-a.e.

(b) $\sum_{k\in\mathbb{Z}} |\,\gamma_{k+1} - \gamma_k\,| + \sum_{k\le 0} |\,\overline{\gamma}_{k+1} - \overline{\gamma}_k\,| < \infty$ and for all $r > 0$ there exists a $\delta_r > 0$ such that $\phi \ge \delta_r$ dx-a.e. on B_r.

(c) (γ) is satisfied, $R_1\big(L^2(\mathbb{R}^d, m)_0\big) \subset C(\mathbb{R}^d)$, and if $\phi \not\equiv 1$ then $R_1(1_G \cdot \rho\sigma_r) \in C(\mathbb{R}^d)$ for any $G \subset \mathbb{R}^d$ relatively compact open and $r > 0$, where σ_r is the surface measure on ∂B_r.

Under the assumptions (a)-(c), we showed (see Theorem 2.1.6) that $\mathbb{M}$ satisfies

$$X_t = x + W_t + \int_0^t \frac{\nabla\rho}{2\rho}(X_s)\, ds + \int_0^\infty \int_0^t \nu_a(X_s)\, d\ell^a_s(\|X\|)\, \eta(da),\ t \ge 0, \tag{3.29}$$

$\mathbb{P}_x$-a.s. for any $x \in \mathbb{R}^d$, where W is a d-dimensional standard Brownian motion starting at 0, ν_a is the unit outward normal on the boundary ∂B_a, $\ell^a(\|X\|)$ is the symmetric semimartingale local time at $a \in (0, \infty)$ of $\|X\|$, $\eta = \sum_{k\in\mathbb{Z}} (2\alpha_k - 1)\delta_{d_k}$ with $(\alpha_k)_{k\in\mathbb{Z}} \subset (0, 1)$ is a sum of Dirac measures at a sequence $(d_k)_{k\in\mathbb{Z}} \subset (0, \infty)$ with exactly two accumulation points in $[0, \infty)$, one is zero and the other is $m_0 > 0$, and $(\alpha_k)_{k\in\mathbb{Z}}$ and $(d_k)_{k\in\mathbb{Z}}$ are determined by $(\gamma_k)_{k\in\mathbb{Z}}$, $(\overline{\gamma}_k)_{k\in\mathbb{Z}}$, $(l_k)_{k\in\mathbb{Z}}$, and $(r_k)_{k\in\mathbb{Z}}$ (see Chapter 2).

Remark 3.2.7. *The assumptions (a)-(c) imply* $(\alpha) - (\delta)$ *and (IBP). However, in comparison to Chapter 2, we insist to point out two improvements. The first one is that in* (α) ρ *is only assumed to be in* $H^{1,1}_{loc}(\mathbb{R}^d)$ *instead of* $\rho = \xi^2$ *with* $\xi \in H^{1,2}_{loc}(\mathbb{R}^d)$ *in (a).* (α) *allows to consider weights that increase rapidly toward singularities which are of positive capacity. A typical example is* $\rho(x) = \|x\|^\alpha$, $\alpha \in (-d+1, -d+2]$ *(cf. [35, Example 3.3.2]). The second improvement is that in* (δ) *the potentials are only assumed to be bounded by continuous functions and not to be continuous as in* (c) *(cf. Proposition 3.1.14 and Lemma 2.3.2, Theorem 2.3.5). In particular, replacing (a) with* (β)*, and (c) with* (γ) *and* (δ)*, we still obtain* (3.29)*.*

Proposition 3.2.8. *(i) Let $\rho(x) = \|x\|^\alpha$, $\alpha \in (-d+1, 2)$. Let ϕ be like in (3.28), satisfy (b) and $\tilde{c}^{-1} \le \phi \le \tilde{c}$ for some $\tilde{c} \ge 1$. If $\phi \equiv \tilde{c}$ dx-a.e. (i.e. $\eta(da) \equiv 0$) or $\phi \not\equiv \tilde{c}$ dx-a.e. and $\alpha \in (-d+1, 1)$, then (3.29) holds.*

(ii) Let $\rho(x) = \|x\|^\alpha$, $\alpha \in [1, d)$. Let ϕ be like in (3.28), satisfy (b) and and $\tilde{c}^{-1} \le \phi \le \tilde{c}$ for some $\tilde{c} \ge 1$. Then (3.29) holds for any $x \in \mathbb{R}^d \setminus \{0\}$.

Proof. (i) The proof follows from Proposition 3.2.3, Lemma 3.2.6 (i), (iv) and (v), and Remark 3.2.7.
(ii) By Lemma 3.2.6 (vi) $(\mathcal{E}^B, D(\mathcal{E}^B))$, $B := \mathbb{R}^d \setminus \{0\}$ satisfies (**H1**), (**H2**). Fix $\alpha \in [1, d)$. Let

$$B_k := \Big\{ x \in \mathbb{R}^d \;\Big|\; \frac{l_{-k+1} + l_{-k}}{2} < \|x\| < \frac{r_{k+1} + r_k}{2} \Big\}, \quad k \ge 1.$$

Then

$$b_k := \tilde{c}^{-1} \Big(\frac{l_{-k+1} + l_{-k}}{2} \Big)^\alpha < \rho\phi < \tilde{c} \, \Big(\frac{r_{k+1} + r_k}{2} \Big)^\alpha =: e_k$$

on B_k. Set $d_k := \max(b_k^{-1}, e_k)$, $k \ge 1$. Then $(B_k)_{k \ge 1}$ is an increasing sequence of relatively compact open sets with smooth boundary such that $\bigcup_{k \ge 1} B_k = \bigcup_{k \ge 1} \overline{B}_k = B$ and $\rho\phi \in (d_k^{-1}, d_k)$ on B_k where $d_k \to \infty$ as $k \to \infty$. Moreover $\|\nabla \rho\| \in L^\infty(B_k, dx)$ for any $k \ge 1$. We may now apply a localization procedure similarly to the one that is presented in all details subsequently to Lemma 3.4.3 to obtain the assertion. We only note that by the Nash type inequality of Lemma 3.4.4 we obtain resolvent kernel estimates as in Corollary 3.4.6. These local resolvent estimates are better than the global ones of Lemma 3.2.6 (ii). □

Skew reflection on a Lipschitz domain

We consider the Dirichlet form determined by (3.13) with $\rho(x) = \|x\|^\alpha$, $\alpha \in (-d+1, d)$ and

$$\phi(x) := \beta \, 1_{G^c}(x) + (1-\beta) 1_G(x) \tag{3.30}$$

where $\beta \in (0, 1)$ and $G \subset \mathbb{R}^d$ is a bounded Lipschitz domain. Then the following integration by parts formula holds for $f \in \{f^1, \dots, f^d\}$, $g \in C_0^\infty(\mathbb{R}^d)$

$$-\mathcal{E}(f, g) = \int_{\mathbb{R}^d} \left(\nabla f \cdot \frac{\nabla \rho}{2\rho} \right) \, g \, dm + (2\beta - 1) \int_{\partial G} \nabla f \cdot \nu \, \frac{\rho}{2} \, d\sigma,$$

where ν denotes the unit outward normal on ∂G (cf. [68] and [70]). The existence of a Hunt process associated to $\mathcal{E}$ that satisfies the absolute continuity condition follows from Lemma 3.2.6 (i). Furthermore:

Theorem 3.2.9. *(i) Let $\alpha \in (-d+1, 1)$. Then*

$$X_t = x + W_t + \alpha \int_0^t \frac{X_s}{2\|X_s\|^2}\, ds + (2\beta - 1) \int_0^t \nu(X_s)\, d\ell_s \quad t \geq 0 \tag{3.31}$$

$\mathbb{P}_x$-a.s. for all $x \in \mathbb{R}^d$, where $(W_t)_{t\geq 0}$ is a d-dimensional Brownian motion starting from zero and $(\ell_t)_{t\geq 0} \in A^+_{c,1}$ is uniquely associated to the surface measure $\frac{\rho\sigma}{2}$ on ∂G via the Revuz correspondence.

(ii) Let $0 \notin \partial G$ and $\alpha \in [1, d)$. Then (3.31) holds $\mathbb{P}_x$-a.s. for any $x \in \mathbb{R}^d \setminus \{0\}$.

Proof. (i) Lemma 3.2.6 (iv) and (v) apply. Therefore (α)-(δ) and (IBP) are satisfied and the assertion immediately follows from Theorem 3.2.4.
(ii) The proof is similar to the proof of Proposition 3.2.8 (ii). We therefore only indicate the sequences $(B_k)_{k\geq 1}$ and $(d_k)_{k\geq 1}$. Fix $\alpha \in [1, d)$ We have either $0 \in G$ or $0 \in \overline{G}^c$. If $0 \in G$, then choose $k_0 \geq 1$ such that $\partial G \subset \{x \in \mathbb{R}^d \mid k_0^{-1} < \|x\| < k_0\}$ and let

$$B_k := \{x \in \mathbb{R}^d \mid (k_0 + k)^{-1} < \|x\| < k_0 + k\}, \quad k \geq 1.$$

Then

$$b_k := \min(\beta, 1-\beta)(k_0 + k)^{-\alpha} < \rho\phi < \max(\beta, 1-\beta)(k_0 + k)^{\alpha} =: e_k$$

and we let $d_k := \max(b_k^{-1}, e_k)$, $k \geq 1$. If $0 \in \overline{G}^c$ then similarly we can find suitable $(B_k)_{k\geq 1}$ and $(d_k)_{k\geq 1}$. □

Remark 3.2.10. *This result is presented in Chapter 2 and generalizes a result obtained by Portenko in [52, III, §3 and §4].*

3.2.2 Skew reflection on hyperplanes

We consider skew reflection on hyperplanes

$$H_s := \{x \in \mathbb{R}^d \mid x_d = s\}, \;\; s \in \mathbb{R}.$$

Let $(l_k)_{k\in\mathbb{Z}} \subset (-\infty, 0)$, $-\infty < l_k < l_{k+1} < 0$ be a sequence converging to 0 as $k \to \infty$ and tending to $-\infty$ as $k \to -\infty$. Let $(r_k)_{k\in\mathbb{Z}} \subset (0, \infty)$, $0 < r_k < r_{k+1} < \infty$ be a sequence converging to 0 as $k \to -\infty$ and tending to infinity as $k \to \infty$. We consider a function

$$\phi(x_d) := \sum_{k \in \mathbb{Z}} \Big(\gamma_{k+1} \cdot 1_{(l_k, l_{k+1})}(x_d) + \overline{\gamma}_{k+1} \cdot 1_{(r_k, r_{k+1})}(x_d) \Big) \tag{3.32}$$

where $\gamma_k, \overline{\gamma}_k \in (0, \infty)$ that only depends on the d-th coordinate. We shall assume

(d) $\rho\phi \in A_2$ and $\rho(x) = \|x\|^\alpha$, $\alpha \in (-d+1, 1)$.

(e) $\sum_{k \geq 0} |\, \gamma_{k+1} - \gamma_k \,| + \sum_{k \leq 0} |\, \overline{\gamma}_{k+1} - \overline{\gamma}_k \,| < \infty$ and $\gamma := \lim_{k \to \infty} \gamma_k$, $\overline{\gamma} := \lim_{k \to -\infty} \overline{\gamma}_k$ are strictly positive.

The assumptions (d), (e) imply (α),(β). Therefore, the closure $(\mathcal{E}, D(\mathcal{E}))$ of (3.13) is a symmetric, regular and strongly local Dirichlet form.

Proposition 3.2.11. *The following integration by parts formula holds for $f, g \in C_0^\infty(\mathbb{R}^d)$*

$$-\mathcal{E}(f,g) = \int_{\mathbb{R}^d} \left(\frac{1}{2} \Delta f + \nabla f \cdot \frac{\nabla \rho}{2\rho} \right) g \; dm + \frac{\overline{\gamma} - \gamma}{2} \int_{\mathbb{R}^d} \partial_d f \, g \, \rho \, \delta_0(dx_d) \, d\bar{x} \tag{3.33}$$
$$+ \sum_{k \in \mathbb{Z}} \frac{\gamma_{k+1} - \gamma_k}{2} \int_{\mathbb{R}^d} \partial_d f \, g \, \rho \, \delta_{l_k}(dx_d) \, d\bar{x} + \sum_{k \in \mathbb{Z}} \frac{\overline{\gamma}_{k+1} - \overline{\gamma}_k}{2} \int_{\mathbb{R}^d} \partial_d f \, g \, \rho \, \delta_{r_k}(dx_d) \, d\bar{x},$$

where $d\bar{x} = dx_1 \cdots dx_{d-1}$. The last two summations in 2nd line are in particular only taken over finitely many negative and positive k, respectively since f has compact support.

Proof. For $f, g \in C_0^\infty(\mathbb{R}^d)$

$$\mathcal{E}(f,g) = \frac{1}{2} \sum_{j=1}^d \int_{\mathbb{R}^d} \partial_j f \partial_j g \; dm$$
$$= -\frac{1}{2} \sum_{j=1}^d \sum_{k \in \mathbb{Z}} \left(\gamma_k \int_{A_k} \left(\partial_{jj} f + \partial_j f \, \frac{\partial_j \rho}{\rho} \right) g \, \rho \, dx \, + \, \overline{\gamma}_k \int_{\hat{A}_k} \left(\partial_{jj} f + \partial_j f \, \frac{\partial_j \rho}{\rho} \right) g \, \rho \, dx \right)$$
$$+ \frac{1}{2} \sum_{j=1}^d \sum_{k \in \mathbb{Z}} \left(\int_{A_k} \gamma_k \, \partial_j \left(\partial_j f \, g \, \rho \right) \, dx + \int_{\hat{A}_k} \overline{\gamma}_k \, \partial_j \left(\partial_j f \, g \, \rho \right) \, dx \right),$$

where $A_k := \mathbb{R}^{d-1} \times (l_{k-1}, l_k)$ and $\hat{A}_k := \mathbb{R}^{d-1} \times (r_{k-1}, r_k)$. The first term equals

$$-\frac{1}{2} \int_{\mathbb{R}^d} \left(\Delta f + \nabla f \cdot \frac{\nabla \rho}{\rho} \right) g \; dm,$$

and the second term equals

$$
\begin{aligned}
& \frac{1}{2}\sum_{k\in\mathbb{Z}} \gamma_k \left(\int_{\mathbb{R}^d} \partial_d f\, g\, \rho\, \delta_{l_k}(dx_d)\, d\bar{x} - \int_{\mathbb{R}^d} \partial_d f\, g\, \rho\, \delta_{l_{k-1}}(dx_d)\, d\bar{x}\right) \\
+ \quad & \frac{1}{2}\sum_{k\in\mathbb{Z}} \overline{\gamma}_k \left(\int_{\mathbb{R}^d} \partial_d f\, g\, \rho\, \delta_{r_k}(dx_d)\, d\bar{x} - \int_{\mathbb{R}^d} \partial_d f\, g\, \rho\, \delta_{r_{k-1}}(dx_d)\, d\bar{x}\right) \\
= \quad - \quad & \frac{1}{2}\left(\sum_{k\in\mathbb{Z}} (\gamma_{k+1} - \gamma_k) \int_{\mathbb{R}^d} \partial_d f\, g\, \rho\, \delta_{l_k}(dx_d)\, d\bar{x} - \lim_{k\to\infty} \gamma_{k+1} \int_{\mathbb{R}^d} \partial_d f\, g\, \rho\, \delta_{l_{k+1}}(dx_d)\, d\bar{x}\right) \\
- \quad & \frac{1}{2}\left(\lim_{k\to-\infty} \overline{\gamma}_k \int_{\mathbb{R}^d} \partial_d f\, g\, \rho\, \delta_{r_{k-1}}(dx_d)\, d\bar{x} + \sum_{k\in\mathbb{Z}} (\overline{\gamma}_{k+1} - \overline{\gamma}_k) \int_{\mathbb{R}^d} \partial_d f\, g\, \rho\, \delta_{r_k}(dx_d)\, d\bar{x}\right) \\
= \quad - \quad & \frac{1}{2}\left(\sum_{k\in\mathbb{Z}} (\gamma_{k+1} - \gamma_k) \int_{\mathbb{R}^d} \partial_d f\, g\, \rho\, \delta_{l_k}(dx_d)\, d\bar{x} - \gamma \int_{\mathbb{R}^d} \partial_d f\, g\, \rho\, \delta_0(dx_d)\, d\bar{x}\right) \\
- \quad & \frac{1}{2}\left(\overline{\gamma} \int_{\mathbb{R}^d} \partial_d f\, g\, \rho\, \delta_0(dx_d)\, d\bar{x} + \sum_{k\in\mathbb{Z}} (\overline{\gamma}_{k+1} - \overline{\gamma}_k) \int_{\mathbb{R}^d} \partial_d f\, g\, \rho\, \delta_{r_k}(dx_d)\, d\bar{x}\right).
\end{aligned}
$$

□

Remark 3.2.12. *The integration by parts formula in Proposition 3.2.11 extends to $f \in \{f^1, \ldots, f^d\}$ and to $f(x) = |f^d(x) - c|$, $c \in \mathbb{R}$.*

Proposition 3.2.13. *There exists a Hunt process $\mathbb{M}$ associated with $(P_t)_{t\geq 0}$, i.e. **(H1)** and **(H2)** hold.*

Proof. Using Proposition 3.2.11 one can see that the functions $f \in C_0^\infty(\mathbb{R}^d)$ satisfying

$$
\begin{aligned}
& \partial_d f(\bar{x}, l_k) = \partial_d f(\bar{x}, r_k) = \partial_d f(\bar{x}, 0) = 0 \quad \text{for all } k \in \mathbb{Z} \\
\text{and} \quad & \frac{1}{2}\Delta f + \nabla f \cdot \frac{\nabla \rho}{2\rho} \in L^2(\mathbb{R}^d, m)
\end{aligned}
\tag{3.34}
$$

are in $D(L)$ where $\bar{x} = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}$. For given $r \in (0, \infty)$, define S_r to be the set of functions $h \in C_0^\infty(\mathbb{R}^d)$ such that

$$
\nabla h(x) = 0, \quad \forall x \in B_r, \qquad \partial_d h(\bar{x}, x_d) = 0 \;\text{ if } -r < x_d < r \tag{3.35}
$$

and h satisfies (3.34). Note that if $h \in S_r$ then h^2 is also in S_r since h^2 satisfies (3.34) and (3.35). Furthermore for $h \in S_r$, $h^2 \in D(L_1)$ by Lemma 3.1.5 (ii). Let

$S = \bigcup_{r\in(0,\infty)} S_r$. Since for $h \in S$

$$Lh \in L^\infty(\mathbb{R}^d, m)_0,$$

$R_1\big([(1-L)h]^+\big)$, $R_1\big([(1-L)h]^-\big)$, $R_1\big([(1-L_1)h^2]^+\big)$, and $R_1\big([(1-L_1)h^2]^-\big)$ are continuous on $\mathbb{R}^d$ by Proposition 3.2.3 (i). Furthermore for all $y \in \mathbb{Q}^d$, $\varepsilon \in \mathbb{Q} \cap (0,1)$ we can find $h \in S$ such that $h \geq 1$ on $\overline{B}_{\frac{\varepsilon}{4}}(y)$, $h \equiv 0$ on $\mathbb{R}^d \setminus B_{\frac{\varepsilon}{2}}(y)$. Therefore, we can find a countable subset $\tilde{S} \subset S$ satisfying $(\mathbf{H2})'$ (i) and (ii). Therefore, by Proposition 3.2.3 (ii) and Lemma 3.1.10 (**H1**) and (**H2**) hold. □

Remark 3.2.14. *By Lemma 3.2.6 (i) we can also show Proposition 3.2.13, i.e. $(P_t)_{t\geq 0}$ is a Feller semigroup*

The assumption

(f) $\tilde{c}^{-1} \leq \phi \leq \tilde{c}$ for some $\tilde{c} \geq 1$

now implies (δ) as in the proof of Proposition 3.2.8. We then obtain the following:

Theorem 3.2.15. *Suppose (d)-(f) and let $\beta := \frac{\overline{\gamma}}{\overline{\gamma}+\gamma}$, $\beta_k := \frac{\gamma_{k+1}}{\gamma_{k+1}+\gamma_k}$, and $\overline{\beta}_k := \frac{\overline{\gamma}_{k+1}}{\overline{\gamma}_{k+1}+\overline{\gamma}_k}$, $k \in \mathbb{Z}$.*

(i) The process $\mathbb{M}$ satisfies

$$X_t^j = x_j + W_t^j + \int_0^t \frac{\partial_j \rho}{2\rho}(X_s)\, ds\ , \quad j = 1, \dots, d-1,$$

$$X_t^d = x_d + W_t^d + \int_0^t \frac{\partial_d \rho}{2\rho}(X_s)\, ds + \int_{\mathbb{R}} \ell_t^a\, \mu(da)\ , \quad t \geq 0 \tag{3.36}$$

$\mathbb{P}_x$ -a.s. for any $x \in \mathbb{R}^d$, where $(W^1, \dots, W^d)$ is a standard d-dimensional Brownian motion starting from zero and

$$\mu := \sum_{k\in\mathbb{Z}} \Big((2\beta_k - 1)\, \delta_{l_k} + (2\overline{\beta}_k - 1)\, \delta_{r_k} \Big) + (2\beta - 1)\, \delta_0 \tag{3.37}$$

where ℓ^{l_k}, ℓ^{r_k} and ℓ^0 are boundary local times of X, i.e. they are positive continuous additive functionals of X in the strict sense associated via the Revuz correspondence (cf. [35, Theorem 5.1.3]) with the weighted surface measures

$\frac{\gamma_{k+1}+\gamma_k}{2}\,\rho\,\delta_{l_k}(dx_d)\,d\bar{x}$ on H_{l_k}, $\frac{\overline{\gamma}_{k+1}+\overline{\gamma}_k}{2}\,\rho\,\delta_{r_k}(dx_d)\,d\bar{x}$ on H_{r_k} and $\frac{\overline{\gamma}+\gamma}{2}\,\rho\,\delta_0(dx_d)\,d\bar{x}$ on H_0 respectively and related via the formulas

$$\begin{aligned}
\mathbb{E}_x\left[\int_0^\infty e^{-t}\,d\ell_t^{l_k}\right] &= R_1\left(\frac{\gamma_{k+1}+\gamma_k}{2}\,\rho\delta_{l_k}(dx_d)\,d\bar{x}\right)(x),\\
\mathbb{E}_x\left[\int_0^\infty e^{-t}\,d\ell_t^{r_k}\right] &= R_1\left(\frac{\overline{\gamma}_{k+1}+\overline{\gamma}_k}{2}\,\rho\delta_{r_k}(dx_d)\,d\bar{x}\right)(x),\\
\mathbb{E}_x\left[\int_0^\infty e^{-t}\,d\ell_t^{0}\right] &= R_1\left(\frac{\overline{\gamma}+\gamma}{2}\,\rho\delta_{0}(dx_d)\,d\bar{x}\right)(x),
\end{aligned}$$

which all hold for any $x \in \mathbb{R}^d$, $k \in \mathbb{Z}$.

(ii) $\big((X_t^d)_{t\geq 0}, \mathbb{P}_x\big)$ is a continuous semimartingale for any $x \in \mathbb{R}^d$ and

$$\mathbb{P}_x\big(\ell_t^a = \ell_t^a(X^d)\big) = 1, \quad \forall x \in \mathbb{R}^d,\ t \geq 0,\ a \in \{0, l_k, r_k : k \in \mathbb{Z}\},$$

where $\ell_t^a(X^d)$ is the symmetric semimartingale local time of X^d at $a \in (-\infty, \infty)$ as defined in [54, VI.(1.25)].

Proof. (i) Since $(\alpha) - (\delta)$ and (IBP) hold, we may apply Theorem 3.2.4. The identification of the drift part follows with the help of Remark 3.2.12. Note that equation (3.36) holds for all $t \geq 0$ since $(\mathcal{E}, D(\mathcal{E}))$ is conservative, see (3.19).
(ii) The first statement is clear from Lemma 3.1.13. In particular, we may apply the symmetric Itô-Tanaka formula (see [54, VI. (1.25)]) and obtain

$$\left|X_t^d - a\right| = |x_d - a| + \int_0^t sign(X_s^d - a)\,dX_s^d + \ell_t^a(X^d), \tag{3.38}$$

$\mathbb{P}_x$ -a.s. for any $x \in \mathbb{R}^d$, $t \geq 0$, where $sign(x) = 1$ for $x > 0$, -1 for $x < 0$ and 0 for $x = 0$ and $\ell_t^a(X^d)$ is the symmetric semimartingale local time of X^d at $a \in (-\infty, \infty)$ as defined in [54, VI.(1.25)]. Let $h_a(x) := |x_d - a|$, $a \in \{0, l_k, r_k : k \in \mathbb{Z}\}$. Then $\partial_d h_a$ is everywhere bounded by one (except in a). Note that for $\partial_d h(x)_a = 1$ for $x_d > a$, -1 and for $x_d < a$ and $\Delta h_a(x) = 0$ for $x_d \neq a$. Let, for instance, $a = l_r$, $r \in \mathbb{Z}$. By Remark 3.2.12 we obtain for $h_{l_r}(x) = \left|x_d - l_r\right|$, $g \in C_0^\infty(\mathbb{R}^d)$

$$
\begin{aligned}
&-\mathcal{E}(h_{l_r}, g)\\
= &\int_{\mathbb{R}^d} \left(\frac{1}{2}\Delta h_{l_r} + \nabla h_{l_r} \cdot \frac{\nabla\rho}{2\rho}\right) g\ dm + \frac{\overline{\gamma}-\gamma}{2}\int_{\mathbb{R}^d} \partial_d h_{l_r}\, g\, \rho\, \delta_0(dx_d)\, d\bar{x}\\
&+\sum_{\substack{k\in\mathbb{Z},\\ k\neq r}} \frac{\gamma_{k+1}-\gamma_k}{2}\int_{\mathbb{R}^d} \partial_d h_{l_r}\, g\, \rho\, \delta_{l_k}(dx_d)\, d\bar{x}\\
&+\sum_{k\in\mathbb{Z}} \frac{\overline{\gamma}_{k+1}-\overline{\gamma}_k}{2}\int_{\mathbb{R}^d} \partial_d h_{l_r}\, g\, \rho\, \delta_{r_k}(dx_d)\, d\bar{x} + \frac{\gamma_{r+1}+\gamma_r}{2}\int_{\mathbb{R}^d} g\, \rho\, \delta_{l_r}(dx_d)\, d\bar{x}\\
= &\int_{\mathbb{R}^d} \frac{\partial_d\rho}{2\rho}(1_{\{x_d>l_r\}} - 1_{\{x_d<l_r\}}) g\ dm + \frac{\overline{\gamma}-\gamma}{2}\int_{\mathbb{R}^d} (1_{\{x_d>l_r\}} - 1_{\{x_d<l_r\}})\, g\, \rho\, \delta_0(dx_d)\, d\bar{x}\\
&+\sum_{\substack{k\in\mathbb{Z},\\ k\neq r}} \frac{\gamma_{k+1}-\gamma_k}{2}\int_{\mathbb{R}^d} (1_{\{x_d>l_r\}} - 1_{\{x_d<l_r\}}) g\, \rho\, \delta_{l_k}(dx_d)\, d\bar{x}\\
&+\sum_{k\in\mathbb{Z}} \frac{\overline{\gamma}_{k+1}-\overline{\gamma}_k}{2}\int_{\mathbb{R}^d} (1_{\{x_d>l_r\}} - 1_{\{x_d<l_r\}})\, g\, \rho\, \delta_{r_k}(dx_d)\, d\bar{x}\\
&+\frac{\gamma_{r+1}+\gamma_r}{2}\int_{\mathbb{R}^d} g\, \rho\, \delta_{l_r}(dx_d)\, d\bar{x}.
\end{aligned}
$$

Thus, applying [35, Theorem 5.5.5] to h_a, which is in $D(\mathcal{E})_{b,loc}$, we obtain again similarly to (i)

$$
\left|X_t^d - a\right| = |x_d - a| + \int_0^t sign(X_s^d - a)\, dX_s^d + \ell_t^a, \tag{3.39}
$$

$\mathbb{P}_x$ -a.s. for any $x \in \mathbb{R}^d$, $t \geq 0$. Comparing (3.38) and (3.39), we get the result. □

Remark 3.2.16. *Similarly to the proofs of Proposition 3.2.8 (ii) and Theorem 3.2.9 (ii), we can also obtain Theorem 3.2.15 for $\alpha \in [1, d)$ but only for all starting points in $\mathbb{R}^d \setminus \{0\}$.*

3.2.3 Further example of A_2 weight that satisfies the absolute continuity condition

In this example, we let $\phi \equiv 1$ and

$$\rho(x) = \begin{cases} ||x||^{\alpha_1} \left| log\, ||x|| \right|^{\alpha_2}, & \text{if } \ ||\text{x}|| \leq 1 \\ ||x||^{\beta_1} \left| log\, ||x|| \right|^{\beta_2}, & \text{if } \ ||\text{x}|| > 1 \end{cases}$$

$\alpha_1 \in (-d+1, d),\ \beta_1 \in (-d, d)$, and $\alpha_2,\ \beta_2 > 0$.

Then $\rho \in H^{1,1}_{loc}(\mathbb{R}^d, dx)$. Moreover, it is known that $\rho \in A_2$ if $\alpha_1, \quad \beta_1 \in (-d, d)$, $\alpha_2, \quad \beta_2 \in \mathbb{R}$ (see [40, Example 1.4]). Therefore, (α) and (β) are satisfied and the closure $(\mathcal{E}, D(\mathcal{E}))$ of (3.13) is a symmetric, regular and strongly local Dirichlet form. Let $(P_t)_{t\geq 0}$ be the transition function defined in Section 3.2 (see Proposition 3.2.3).

Proposition 3.2.17. *There exists a Hunt process $\mathbb{M}$ associated with $(P_t)_{t\geq 0}$, i.e.* ***(H1)*** *and* ***(H2)*** *hold.*

Proof. By (3.33) the functions $f \in C_0^\infty(\mathbb{R}^d)$ satisfying

$$\Delta f + \nabla f \cdot \frac{\nabla \rho}{\rho} \in L^2(\mathbb{R}^d, m)$$

are in $D(L)$. Since

$$\frac{\nabla \rho}{\rho}(x) = \begin{cases} \alpha_1 x\ \|x\|^{-2} + \alpha_2 x\ \|x\|^{-1}\ \left| log\|x\| \right|^{-1}, & \text{if } \ ||\text{x}|| \leq 1, \\ \beta_1 x\ \|x\|^{-2} + \beta_2 x\ \|x\|^{-1}\ \left| log\|x\| \right|^{-1}, & \text{if } \ ||\text{x}|| > 1, \end{cases}$$

we can find $h \in D(L)$ such that

$$\begin{aligned} h \in C_0^\infty(\mathbb{R}^d), \quad \nabla h(0) = 0, \quad \nabla h(x) = 0,\ x \in \partial B_1, \\ \text{and } \ Lh = \frac{1}{2}\ \Delta h + \nabla h \cdot \frac{\nabla \rho}{2\rho} \in L^\infty(\mathbb{R}^d, m)_0. \end{aligned} \tag{3.40}$$

Define S to be the set of functions $h \in C_0^\infty(\mathbb{R}^d)$ satisfying (3.40). Clearly, if $h \in S$, then $h^2 \in S$. Furthermore for $h \in S$, $h^2 \in D(L_1)$ by Lemma 3.1.5 (ii). Therefore, for $h \in S$ $R_1\big([(1-L)h]^+\big)$, $R_1\big([(1-L)h]^-\big)$, $R_1\big([(1-L_1)h^2]^+\big)$, and $R_1\big([(1-L_1)h^2]^-\big)$ are continuous on $\mathbb{R}^d$ by Proposition 3.2.3 (i). Then, we can show that there exists a Hunt process $\mathbb{M}$ associated with $(P_t)_{t\geq 0}$ similarly to Proposition 3.2.13. $\square$

Remark 3.2.18. *Since the transition density estimate* (3.2.2) *is not explicitly calculated, it may be difficult to show that $(P_t)_{t\geq 0}$ is a Feller semigroup. However we apply Dirichlet form method to show Proposition 3.2.17.*

3.3 Weakly differentiable weights with moderate growth at singularities

Let $d \geq 2$. In this section we shall assume

(ε) $\rho \in H^{1,1}_{loc}(\mathbb{R}^d, dx)$, $\rho > 0$ dx-a.e.

(ζ) $\frac{\|\nabla \rho\|}{\rho} \in L^{d+\varepsilon}_{loc}(\mathbb{R}^d, m)$ for some $\varepsilon > 0$, $m := \rho dx$.

Remark 3.3.1. *(i) (ε) and (ζ) are equivalent to (H1) and (H2) in [6, p.2].*
(ii) The order of integrability of the logarithmic derivative $\frac{\|\nabla \rho\|}{\rho}$ tells us how fast it grows at its singularities $\{\rho = 0\}$.

We consider the symmetric positive definite bilinear form

$$\mathcal{E}(f, g) := \frac{1}{2}\int_{\mathbb{R}^d} \nabla f \cdot \nabla g \; dm, \quad f, g \in C_0^\infty(\mathbb{R}^d). \tag{3.41}$$

(ε) implies that (3.41) is closable in $L^2(\mathbb{R}^d, m)$. The closure $(\mathcal{E}, D(\mathcal{E}))$ of (3.41) is a regular, strongly local, symmetric Dirichlet form. By [6, Corollary 2.2] ρ has a Hölder continuous version on $\mathbb{R}^d$ that we denote by ρ again. In particular,

$$E := \{x \in \mathbb{R}^d \mid \rho(x) > 0\}$$

is open in $\mathbb{R}^d$. We can hence consider the part Dirichlet form $(\mathcal{E}^E, D(\mathcal{E}^E))$ of $(\mathcal{E}, D(\mathcal{E}))$ on E (see Section 3.1). Moreover, by [6, Theorem 1.1, Proposition 3.2] there exists a Hunt process

$$\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \zeta, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$$

with transition kernel $p_t(x, dy)$ (from E to E) and transition kernel density $p_t(\cdot,\cdot) \in \mathcal{B}(E \times E)$, i.e. $p_t(x, dy) = p_t(x, y)\; m(dy)$, such that for $f \in L^{d+\varepsilon}(E, m)$

$$P_t f(x) := \int f(y)\, p_t(x, y)\; m(dy), \quad x \in E$$

is in $C(E)$ and $P_t f = T_t f$ m-a.e. Note that $p_t(\cdot,\cdot)$ can be defined on $E \times E$ since $\mathrm{Cap}(\mathbb{R}^d \setminus E) = 0$ (see [6, Proposition 3.2, Lemma 4.1]).

Lemma 3.3.2. *Let $f \in \mathcal{B}_b(E)$ with compact support, i.e.* $\mathrm{supp}(|f|m)$ *is compact. Then $P_t f$ is an m-version of $T_t^E f$.*

Proof. Let $\overline{\mathbb{M}} = \left((\overline{X}_t)_{t\geq 0}, (\overline{\mathbb{P}}_x)_{x\in\mathbb{R}^d}\right)$ be the Hunt process associated with regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ and $\overline{\mathbb{M}}|_E = \left((\overline{X}_t^E)_{t\geq 0}, (\overline{\mathbb{P}}_x)_{x\in E}\right)$ be the Hunt process associated with the regular Dirichlet form $(\mathcal{E}^E, D(\mathcal{E}^E))$ (cf. [35, Chapter 7]). Then for any $f \in \mathcal{B}_b(E)$ with compact support and m-a.e. $x \in E$

$$\begin{aligned}
T_t^E f(x) &= \overline{\mathbb{E}}_x[f(\overline{X}_t^E),\ t < \sigma_{E^c}] = \overline{\mathbb{E}}_x[f(\overline{X}_t),\ t < \sigma_{E^c}] = \overline{\mathbb{E}}_x[f(\overline{X}_t)] = T_t f(x) \\
&= \int f(y)\, P_t(x, dy),
\end{aligned}$$

where the second equality follows from the definition of part process and the third since $\mathrm{Cap}(\mathbb{R}^d \setminus E) = 0$ (cf. [44, Proposition 5.30 (i)]) and the last since f is in particular in $L^{d+\varepsilon}(E, m)$. □

By Lemma 3.3.2 the Hunt process $\mathbb{M}$ is associated with $(\mathcal{E}^E, D(\mathcal{E}^E))$ and satisfies the absolute continuity condition. For $f \in \{f^1, \dots f^d\}$ and $g \in C_0^\infty(E)$:

$$-\,\mathcal{E}^E(f, g) = \int_E \left(\nabla f \cdot \frac{\nabla\rho}{2\rho}\right) g\ dm. \tag{3.42}$$

Theorem 3.3.3. *Let $f \in L^{d+\varepsilon}_{loc}(E, m)$ for some $\varepsilon > 0$ and G be any relatively compact open set in E. Then, $1_G \cdot |f| m \in S_{00}$. In particular $1_G \cdot |\partial_j \rho|\, dx \in S_{00}$, $j = 1, \dots, d$.*

Proof. Since $1_G \cdot f \in L^{d+\varepsilon}(E, m)$ for some $\varepsilon > 0$, we get $R_1(1_G \cdot |f|) \in C(E)$ by [6, Proposition 3.5 (iii)]. The assertion now follows by Proposition 3.1.14. □

Theorem 3.3.4. *It holds $\mathbb{P}_x$ -a.s. for any $x \in E$*

$$X_t = x + W_t + \int_0^t \frac{\nabla\rho}{2\,\rho}(X_s)\, ds, \quad t < \zeta, \tag{3.43}$$

where W is a standard d-dimensional Brownian motion on E and ζ is the life time of X.

Proof. Applying [35, Theorem 5.5.5] to $(\mathcal{E}^E, D(\mathcal{E}^E))$ the result follows by (3.42), Theorem 3.3.3, and (3.12). □

Remark 3.3.5. *If $(\mathcal{E}, D(\mathcal{E}))$ is conservative, then* (3.43) *holds with ζ replaced by ∞.*

3.4 Weakly differentiable weights and normal reflection

In this section we show that the Skorokhod decomposition of [68] can be obtained pointwise in the symmetric case, i.e. the non-sectorial perturbation B that is considered in [68] is assumed to be identically zero here. We rely on some results of [11] (cf. Remark 3.4.2).

Let $G \subset \mathbb{R}^d$, $d \geq 2$ be a relatively compact open set with Lipschitz boundary ∂G. Let $\rho \in H^{1,1}(G, dx)$, $\rho > 0$ dx-a.e. and let $m := \rho dx$. Then by [68, Lemma 1.1 (ii)]

$$\mathcal{E}(f, g) := \frac{1}{2} \int_G \nabla f \cdot \nabla g \; dm, \quad f, g \in C^\infty(\overline{G})$$

is closable in $L^2(G, m)$. The closure $(\mathcal{E}, D(\mathcal{E}))$ is a regular, strongly local and conservative Dirichlet form (see [68]).

The following lemma holds also under more general assumptions than the ones that we present. But these are sufficient and suitable for us.

Lemma 3.4.1. *Suppose that $\rho = \xi^2$, $\xi \in H^{1,2}(G, dx)$, $\rho > 0$ dx-a.e. and that $\rho \in C(\overline{G})$. Then*

(i) It holds $\mathrm{Cap}(\overline{G} \cap \{\rho = 0\}) = 0$.

(ii) Let

$$\mathcal{E}(f, g) := \frac{1}{2} \int_G \nabla f \cdot \nabla g \; dm, \quad f, g \in \mathcal{D},$$

where

$$\mathcal{D} := \{f \in C(\overline{G}) \cap H^{1,1}_{loc}(G, dx) \mid \mathcal{E}(f, f) < \infty\}.$$

Then $(\mathcal{E}, \mathcal{D})$ is closable in $L^2(G, m)$ and its closure $(\mathcal{E}, \overline{\mathcal{D}})$ is equal to $(\mathcal{E}, D(\mathcal{E}))$.

Proof. (i) Defining $\xi_\varepsilon := \max(|\xi|, \varepsilon)$ and $f_\varepsilon := -\log(\xi_\varepsilon)$ for $\varepsilon > 0$ the proof is nearly identical to the proof of [31, Theorem 2]. Since $\xi_\varepsilon \in H^{1,2}(G, dx)$, $f_\varepsilon \in H^{1,2}(G, dx)$. Therefore we can choose $g_k \in C^\infty(\overline{G}) \subset D(\mathcal{E})$, $k \in \mathbb{N}$ such that $g_k \to f_\varepsilon$ and $\nabla g_k \to \nabla f_\varepsilon$ as $k \to \infty$ in $L^2(G, dx)$ (cf. [25, Theorem 3, Section 4.2]). Since ρ is bounded above on $\overline{G}$, $g_k \to f_\varepsilon$ and $\nabla g_k \to \nabla f_\varepsilon$ in $L^2(G, m)$ as $k \to \infty$. Hence

$$\sup_{k \in \mathbb{N}} \mathcal{E}(g_k, g_k) < \infty,$$

and so $f_\varepsilon \in D(\mathcal{E})$ by [44, Ch. 1, Lemma 2.12]. Now we can follow the proof of [31, Theorem 2] and [28, Theorem 4.5 (i)] to conclude it.
(ii) Clearly, $C^\infty(\overline{G}) \subset \mathcal{D}$. Let $f \in \mathcal{D}$. Since $\mathrm{Cap}(\overline{G} \cap \{\rho = 0\}) = 0$, there exist open sets $U_j \supset \overline{G} \cap \{\rho = 0\}$ and $\phi_j \in D(\mathcal{E})$ such that $0 \le \phi_j \le 1$ m-a.e., $\phi_j = 1$ m-a.e. on U_j, $j \in \mathbb{N}$, and

$$\lim_{j\to\infty} \int_{\overline{G}} \left(\|\nabla \phi_j\|^2 + |\phi_j|^2 \right) \, dm = 0. \tag{3.44}$$

Define $f_j := f(1-\phi_j)$. There exists a subsequence, denoted by f_j again, such that

$$\lim_{j\to\infty} \int_{\overline{G}} \left(\|\nabla (f_j - f)\|^2 + |f_j - f|^2 \right) \, dm = \lim_{j\to\infty} \int_{\overline{G}} \left(\|\phi_j \, \nabla f + f \, \nabla \phi_j\|^2 + |f \, \phi_j|^2 \right) \, dm = 0$$

by (3.44). Therefore it suffices to find $(f_j^n)_{n\ge 1} \subset C^\infty(\overline{G})$ such that $f_j^n \to f_j$ and $\nabla f_j^n \to \nabla f_j$ in $L^2(G,m)$ as $n \to \infty$. We present two methods to prove this.

1) Observe that $f_j \in H^{1,2}(G, dx)$ since ρ is bounded above and away from zero on $\overline{G} \setminus U_j$ and since $\mathrm{supp} f_j \subset \overline{G} \setminus U_j$. By [25, Theorem 3, Section 4.2], there exists $(f_j^n)_{n\ge1} \subset C^\infty(\overline{G})$ such that $f_j^n \to f_j$ and $\nabla f_j^n \to \nabla f_j$ in $L^2(G,dx)$ as $n \to \infty$. This implies that $f_j^n \to f_j$ and $\nabla f_j^n \to \nabla f_j$ in $L^2(G,m)$ as $n \to \infty$ because ρ is bounded above on $\overline{G}$.

2) Define for $\varepsilon > 0$, $f_j^\varepsilon(x) := \int_\Omega \eta_\varepsilon(x-y)\, f_j(y)\, dy$, $x \in \mathbb{R}^d$ where $\eta_\varepsilon(x) := \frac{1}{\varepsilon^d}\eta\left(\frac{x}{\varepsilon}\right)$, $x \in \mathbb{R}^d$ with $\eta \in C^\infty(\overline{B}_1)$ and $\int_{\mathbb{R}^d} \eta(x)\, dx = 1$. Then $f_j^\varepsilon \in C^\infty(\overline{G})$. Let V and W be the open sets such that $V \subset \overline{V} \subset W \subset \overline{W} \subset G$. For $x \in V$ and sufficiently small $\varepsilon > 0$,

$$\begin{aligned} |\, f_j^\varepsilon(x) \,| &= \left| \int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)\, f_j(y)\, dy \right| = \left| \int_{B_\varepsilon(x) \cap G \setminus U_j} \eta_\varepsilon(x-y)\, f_j(y)\, dy \right| \\ &\le \left(\int_{B_\varepsilon(x) \cap G \setminus U_j} \eta_\varepsilon(x-y)\, \frac{1}{\rho(y)}\, dy \right)^{1/2} \left(\int_{B_\varepsilon(x) \cap G \setminus U_j} \eta_\varepsilon(x-y)\, |\, f_j(y)\,|^2\, \rho(y)\, dy \right)^{1/2} \end{aligned}$$

Since $\int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)\, dy = 1$ and ρ is bounded below on $\overline{G} \setminus U_j$, the last inequlaity implies that

$$\int_V |\, f_j^\varepsilon(x)\,|^2\,\rho(x)\,dx \le c \int_V \left(\int_{B_\varepsilon(x)\cap G\setminus U_j} \eta_\varepsilon(x-y)\,|\, f_j(y)\,|^2\,\rho(y)\,dy \right) \rho(x)\,dx$$
$$\le\ C \int_W |\, f_j(y)\,|^2 \left(\int_{B_\varepsilon(y)} \eta_\varepsilon(x-y)\ \rho(x)\ dx \right) \rho(y)\ dy$$
$$\le C\ \sup_{x\in\overline{G}} \rho(x) \int_W |\ f_j(y)\ |^2 \rho(y)\ dy,$$

where $C = \frac{1}{\inf_{x\in\overline{\Omega}\setminus U_j}\rho(x)}$. The second inequality holds since $(B_\varepsilon(x)\cap G\setminus U_j)\subset W$ for $x\in V$ and sufficiently small $\varepsilon>0$. Similarly, since for $f_j\in H^{1,2}(G,\,dx)$ and sufficiently small $\varepsilon>0$, $\partial_i f_j^\varepsilon = (\partial_i f_j)^\varepsilon$, dx-a.e. on V, it follows

$$\int_V |\,\partial_i f_j^\varepsilon(x)\,|^2\,\rho(x)\,dx \le\ c \int_W |\,\partial_i f_j(y)\,|^2 \rho(y)\,dy.$$

Let $z\in\partial G$. Then since G is Lipschitz, there exists $B_r(z)$, $r>0$ such that for small inward normal vector h at z with $\big(B_r(z)\cap\overline{G}\big)+h\subset\overline{\big(B_r(z)\cap\overline{G}\big)+h}\subset G$,

$$\lim_{h\to 0}\int_{B_r(z)\cap\overline{G}} |\, f_j(x+h)-f_j(x)\,|^2\,\rho(x)\,dx$$
$$\le\ \lim_{h\to 0}\ \sup_{x\in\overline{G}}\rho(x)\int_{B_r(z)\cap\overline{G}} |\, f_j(x+h)-f_j(x)\,|^2\,dx = 0.$$

Similarly,

$$\lim_{h\to 0}\int_{B_r(z)\cap\overline{G}} |\,\partial_i f_j(x+h)-\partial_i f_j(x)\,|^2\,\rho(x)\ dx = 0.$$

Now we can follow the proof of [24, Theorem 3, Section 5.3.3] in order to conclude that for each $j\in\mathbb{N}$ there exists a sequence $\{f_j^n\}_{n\in\mathbb{N}}\subset C^\infty(\overline{G})$ such that $f_j^n\to f_j$ and $\nabla f_j^n\to\nabla f_j$ in $L^2(G,m)$ as $n\to\infty$. □

From now on, we assume

(η) $\rho=\xi^2$, $\xi\in H^{1,2}(G,dx)$, $\rho\in C(\overline{G})$ (and $\rho>0$ dx-a.e. on the bounded Lipschitz domain G)

and

(θ) There exists an open set $E \subset \overline{G}$ with $\mathrm{Cap}(\overline{G} \setminus E) = 0$ such that $(\mathcal{E}, D(\mathcal{E}))$ satisfies the absolute continuity condition on E.

By (θ), we mean that there exists a Hunt process

$$\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in E})$$

with transition kernel $p_t(x, dy)$ (from E to E) and transition kernel density $p_t(\cdot, \cdot) \in \mathcal{B}(E \times E)$, i.e. $p_t(x, dy) = p_t(x, y)\ m(dy)$, such that

$$P_t f(x) := \int f(y)\ p_t(x, y)\ m(dy), \quad t > 0,\ x \in E,\ f \in \mathcal{B}_b(E)$$

with trivial extension to $\overline{G}$ is an m-version of $T_t^{\overline{G}} f$ for any $f \in \mathcal{B}_b(E)$, and $(T_t^{\overline{G}})_{t>0}$ denotes the semigroup associated to $(\mathcal{E}, D(\mathcal{E}))$. In particular $\mathbb{M}$ is a conservative diffusion on E as in (3.19) and (3.20).

Remark 3.4.2. *Lemma 3.4.1 (ii) shows that the Dirichlet form that is considered in [27], [28], and in [11] in case of bounded G is a special case of the generalized Dirichlet form for which an explicit Skorokhod decomposition is derived in [68] for q.e. starting point. In [11] also unbounded Lipschitz domains are considered and according to [11, Theorem 1.14] (θ) holds with $E = (G \cup \Gamma_2) \cap \{\rho > 0\}$ where Γ_2 is an open subset of ∂G that is locally C^2-smooth, provided $\frac{\|\nabla \rho\|}{\rho} \in L^p_{loc}(\overline{G} \cap \{\rho > 0\}, m)$ for some $p \geq 2$ with $p > \frac{d}{2}$ and $\mathrm{Cap}(\overline{G} \setminus E) = 0$.*

Since E is open in $\overline{G}$, we can consider the part Dirichlet form $(\mathcal{E}^E, D(\mathcal{E}^E))$ of $(\mathcal{E}, D(\mathcal{E}))$ on E (see Section 3.1). Now exactly as in Lemma 3.3.2, we show the following lemma.

Lemma 3.4.3. *Let $f \in \mathcal{B}_b(E)$. Then $P_t f$ is an m-version of $T_t^E f$.*

By Lemma 3.4.3 the Hunt process $\mathbb{M}$ is associated with $(\mathcal{E}^E, D(\mathcal{E}^E))$ and satisfies the absolute continuity condition.

In addition to (η) and (θ), we assume

(ι) There exists an increasing sequence of relatively compact open sets $\{B_k\}_{k \in \mathbb{N}} \subset E$ such that $\partial B_k,\ k \in \mathbb{N}$ is Lipschitz, $\bigcup_{k \geq 1} B_k = E$ and $\rho \in (d_k^{-1}, d_k)$ on B_k where $d_k \to \infty$ as $k \to \infty$.

According to [68] the closure of

$$\mathcal{E}^{\overline{B}_k}(f,g) = \frac{1}{2}\int_{B_k} \nabla f \cdot \nabla g \; dm, \quad f,g \in C^\infty(\overline{B}_k),$$

in $L^2(\overline{B}_k, m) \equiv L^2(B_k, m)$, $k \geq 1$, denoted by $(\mathcal{E}^{\overline{B}_k}, D(\mathcal{E}^{\overline{B}_k}))$, is a regular conservative Dirichlet form on $\overline{B}_k$ and moreover, it holds:

Lemma 3.4.4. *(Nash type inequality) Let B_k be as in (ι) and $k \in \mathbb{N}$.*

(i) If $d \geq 3$, then for $f \in D(\mathcal{E}^{\overline{B}_k})$

$$\|f\|_{2,B_k}^{2+\frac{4}{d}} \leq c_k \left[\mathcal{E}^{\overline{B}_k}(f,f) + \|f\|_{2,B_k}^2\right] \|f\|_{1,B_k}^{\frac{4}{d}}. \tag{3.45}$$

(ii) If $d = 2$, then for $f \in D(\mathcal{E}^{\overline{B}_k})$ and any $\delta > 0$

$$\|f\|_{2,B_k}^{2+\frac{4}{d+\delta}} \leq c_k \left[\mathcal{E}^{\overline{B}_k}(f,f) + \|f\|_{2,B_k}^2\right] \|f\|_{1,B_k}^{\frac{4}{d+\delta}}. \tag{3.46}$$

Here $c_k > 0$ is a constant which goes to infinity as $k \to \infty$.

Proof. (i) Let $\varepsilon \in (0,1)$. For $f \in D(\mathcal{E}^{\overline{B}_k})$,

$$\begin{aligned}\int_{B_k} f^2(x)\; \rho(x)\; dx &= \int_{B_k} |f|^{2-\varepsilon}(x)\; |f|^\varepsilon(x)\; \rho(x)\; dx \\ &\leq \left(\int_{B_k} |f(x)|^{\frac{2-\varepsilon}{1-\varepsilon}}\; \rho(x)\; dx\right)^{1-\varepsilon} \left(\int_{B_k} |f(x)|\, \rho(x)\, dx\right)^{\varepsilon}.\end{aligned}$$

By Sobolev's inequality on B_k (cf. e.g. [1, Theorem 4.12 Case C]) and the fact that ρ is bounded above and away from zero on B_k,

$$\|f\|_{2,B_k}^2 \leq c\, d_k^{\frac{4-3\varepsilon}{2}} \left(\int_{B_k} \|\nabla f(x)\|^2\, \rho(x)\, dx + \int_{B_k} |f(x)|^2\, \rho(x)\, dx\right)^{\frac{2-\varepsilon}{2}} \|f\|_1^\varepsilon$$

where $2 \leq \frac{2-\varepsilon}{1-\varepsilon} \leq \frac{2d}{d-2}$ and d_k is as in (ι). Therefore,

$$\|f\|_{2,B_k}^{2+\frac{2\varepsilon}{2-\varepsilon}} \leq c^{\frac{2}{2-\varepsilon}} d_k^{\frac{4-3\varepsilon}{2-\varepsilon}} \left(\int_{B_k} \|\nabla f(x)\|^2\, \rho(x)\, dx + \int_{B_k} |f(x)|^2\, \rho(x)\, dx\right) \|f\|_1^{\frac{2\varepsilon}{2-\varepsilon}}$$

Setting $\varepsilon = \frac{4}{d+2}$, the assertion follows.
(ii) The proof is same as in (i) except that we set $\varepsilon = \frac{4}{d+2+\delta}$ where $\delta > 0$ is arbitrary

and that we use the Sobolev's inequality for $d = 2$ (cf. e.g. [1, Theorem 4.12 Case B]). □

Proposition 3.4.5. *We have for m-a.e. $x, y \in B_k$:*

(i) If $d \geq 3$, the transition kernel density $p_t^{B_k}(\cdot,\cdot)$ has the following upper bound

$$p_t^{B_k}(x,y) \leq C c_k^{d/2} t^{-d/2} \exp\left(t + \frac{-\|x-y\|^2}{8t}\right),$$

where c_k is the constant in (3.45) and $C \in (0,\infty)$ depends on d.

(ii) If $d = 2$ and $\delta > 0$,

$$p_t^{B_k}(x,y) \leq C c_k^{(d+\delta)/2} t^{-(d+\delta)/2} \exp\left(t + \frac{-\|x-y\|^2}{8t}\right),$$

where c_k is the constant in (3.46) and $C \in (0,\infty)$ depends only on $d+\delta$.

Proof. (i) By [38, Section 3] and [19, (2.1)] the $L^2(\overline{B}_k, m)$-semigroup $(T_t^{\overline{B}_k})_{t>0}$ of $\mathcal{E}^{\overline{B}_k}$ admits a heat kernel $p_t^{\overline{B}_k}(x,y)$ which is unique for m-a.e. $x, y \in \overline{B}_k$. By [19, (3.25)], we then have for m-a.e. $x, y \in \overline{B}_k$ that for some constant $C = C(d) \in (0,\infty)$

$$p_t^{\overline{B}_k}(x,y) \leq C\left(\frac{c_k}{t}\right)^{d/2} \exp\left(t - |\psi(x)-\psi(y)| + 2t\|\nabla\psi\|^2_{\infty,\overline{B}_k}\right), \quad t > 0 \tag{3.47}$$

for any $\psi \in C^\infty(\overline{B}_k)$, c_k is the constant in (3.45). Choose $x_0, y_0 \in \overline{B}_k$ as above and let

$$\psi(x) = \left(\frac{x_0 - y_0}{4t}\right)\cdot x, \quad x \in \overline{B}_k.$$

Then

$$p_t^{\overline{B}_k}(x,y) \leq C\left(\frac{c_k}{t}\right)^{d/2} \exp\left(t - \frac{\|x_0-y_0\|^2}{8t}\right). \tag{3.48}$$

Since $(\mathcal{E}^{B_k}, D(\mathcal{E}^{B_k}))$ is the part Dirichlet form of $(\mathcal{E}^{\overline{B}_k}, D(\mathcal{E}^{\overline{B}_k}))$, it is easy to see that

$$p_t^{B_k}(x,y) \leq p_t^{\overline{B}_k}(x,y) \quad \text{for } m\text{-a.e. } x, y \in B_k. \tag{3.49}$$

Now combining (3.48) and (3.49) the assertion follows.
(ii) The proof of (ii) is the same as (i) by using (3.46). □

Corollary 3.4.6. *We have for m-a.e. $x, y \in B_k$*

(i) if $d \geq 3$, then

$$r_1^{B_k}(x,y) \leq c\frac{1}{\|x-y\|^{d-2}}.$$

(ii) if $d = 2$, then for any $\delta > 0$

$$r_1^{B_k}(x,y) \leq c\frac{1}{\|x-y\|^{d+\delta-2}}.$$

Proof. Follows from Proposition 3.4.5 by standard calculations. □

Lemma 3.4.7. *The following integration by parts formula holds for $f \in \{f^1, \ldots, f^d\}$ and $g \in C_0^\infty(B_k)$:*

$$-\mathcal{E}^{B_k}(f,g) = \frac{1}{2}\int_{B_k}\left(\nabla f \ \cdot \frac{\nabla\rho}{\rho}\right) g \ dm + \frac{1}{2}\int_{B_k\cap\partial G} \nabla f\cdot\eta \ g \ \rho \ d\sigma,$$

where η is a unit inward normal vector on $B_k \cap \partial G$ and σ is the surface measure on ∂G.

Proof. See [68, proof of Theorem 5.4]. □

Lemma 3.4.8. *(i) $1_{B_k\cap\partial G}\cdot\rho\sigma \in S_{00}^{B_k}$.*
(ii) Let $f \in L^{\frac{d}{2}+\varepsilon}(B_k, dx)$ for some $\varepsilon > 0$. Then

$$1_{B_k}\cdot|f|m \in S_{00}^{B_k}.$$

In particular $1_{B_k}\cdot\|\nabla\rho\|dx \in S_{00}^{B_k}$ for $d = 2, 3$ and for $d \geq 4$, if $\|\nabla\rho\| \in L^{\frac{d}{2}+\varepsilon}(B_k, dx)$ for some $\varepsilon > 0$.

Proof. (i) Let $d \geq 3$. For m-a.e. $x \in B_k$ by Corollary 3.4.6

$$R_1^{B_k}(1_{B_k\cap\partial G}\cdot\rho\sigma)(x) \leq \sup_{y\in B_k}\rho(y)\int_{\partial G}\frac{1}{\|x-y\|^{d-2}}\sigma(dy).$$

Since $1_{B_k\cap\partial G}\cdot\rho\sigma$ is a positive Radon measure and since the last term is continuous on $\mathbb{R}^d$ by Lemma 3.2.5 (cf. proof of Lemma 3.2.6 (v)), the assertion follows from Proposition

3.1.14 with E replaced by B_k. The proof for $d = 2$ is similar.
(ii) $1_{B_k} \cdot |f|m$ is a positive finite measure on B_k and for m-a.e. $x \in B_k$

$$R_1^{B_k}(1_{B_k} \cdot |f|m)(x) \leq \sup_{y \in B_k} \rho(y) V_\eta(1_{B_k} \cdot |f|)(x)$$

by Corollary 3.4.6 where $\eta = 2 - \delta$ if $d = 2$ and $\eta = 2$ if $d \geq 3$. The assertion now follows from Lemma 3.2.5 and Proposition 3.1.14. We can also prove this by direct calculation as follows. Let $d \geq 3$. For m-a.e. $x \in B_k$

$$\begin{aligned} R_1^{B_k}(1_{B_k} \cdot |f|m)(x) &= \int_{B_k} r_1^{B_k}(x,y)\, |f(y)|\; m(dy) \\ \leq\ & c \int_{B_k} \frac{1}{\|x-y\|^{d-2}}\, |f(y)|\; m(dy) \\ \leq\ & c \sup_{y \in B_k} \rho(y) \left(\int_{B_k} |f(y)|^\alpha\; m(dy) \right)^{1/\alpha} \left(\int_{B_k} \left(\frac{1}{\|x-y\|^{d-2}} \right)^{\alpha^*} dy \right)^{1/\alpha^*}, \end{aligned}$$

where $1/\alpha + 1/\alpha^* = 1$. The first inequality holds by Corollary 3.4.6. The second part of the last term is finite independently of $x \in B_k$ if and only if $1 < \alpha^* < \frac{d}{d-2}$ which is equivalent to $\frac{d}{2} < \alpha < \infty$. The proof for $d = 2$ is similar.
□

In view of Lemma 3.4.8 (ii), we assume from now on

(κ) If $d \geq 4$ and $k \geq 1$, then $\|\nabla \rho\| \in L^{\frac{d}{2}+\varepsilon_k}(B_k, dx)$ for some $\varepsilon_k > 0$.

Proposition 3.4.9. *The process $\mathbb{M}$ satisfies*

$$X_t = x + W_t + \int_0^t \frac{\nabla \rho}{2\,\rho}(X_s)\, ds + \int_0^t \eta(X_s)\, d\ell_s^k \quad t < \sigma_{B_k^c} \tag{3.50}$$

$\mathbb{P}_x$-a.s. for any $x \in B_k$ where W is a standard d-dimensional Brownian motion starting from zero and ℓ^k is the positive continuous additive functional of X^{B_k} in the strict sense associated via the Revuz correspondence (cf. [35, Theorem 5.1.3]) with the weighted surface measure $\frac{1}{2}\rho\sigma$ on $B_k \cap \partial G$.

Proof. We apply [35, Theorem 5.5.5] to $(\mathcal{E}^{B_k}, D(\mathcal{E}^{B_k}))$. By Lemmas 3.4.7, 3.4.8, (3.12) and the Revuz correspondence (cf. [35, Theorem 5.1.3]), the assertion follows (see Theorem 3.2.4 for details). □

Lemma 3.4.10. *$\mathbb{P}_x\big(\lim_{k\to\infty}\sigma_{B_k^c}=\infty\big)=1$ for all $x\in E$.*

Proof. By definition $\{B_k\}_{k\geq 1}$ is an increasing sequence of relatively open sets with $\bigcup_{k\geq 1} B_k = E$. The Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is strongly local and conservative. Hence $\mathbb{P}_x\big(\lim_{k\to\infty}\sigma_{B_k^c}=\infty\big)=1$ for all $x\in \overline{G}\setminus N$ by [35, Lemma 5.5.2 (ii)] where N is an exceptional set. Since N is an exceptional set, $u(x):=\mathbb{P}_x\big(\sigma_N<\infty\big)=0$, m-a.e. x. Furthermore, since u is an excessive function and the resolvent kernel $R_\alpha^E(x,\cdot)$ is absolutely continuous with respect to m for each $\alpha>0$ and $x\in E$, $u(x)=\lim_{\alpha\to\infty}\alpha R_\alpha^E u(x)=0$ for all $x\in E$. Let $x\in E=\bigcup_{k\geq 1}B_k$. Then $x\in B_{k_0}$ for some $k_0\in\mathbb{N}$. This implies that

$$\mathbb{P}_x(\Omega_1)=1,$$

where $\Omega_1:=\{\omega\in\Omega \mid \sigma_{B_{k_0}^c}(\omega)>0\}$. For $\omega\in\Omega_1$, $\forall k\geq k_0$, and small $t=t(\omega)>0$

$$\sigma_{B_k^c}(\omega)\circ\theta_t\leq\sigma_{B_k^c}(\omega).$$

Therefore, for $\omega\in\Omega_1$

$$\lim_{t\to 0}\lim_{k\to\infty}\sigma_{B_k^c}(\omega)\circ\theta_t\leq\lim_{k\to\infty}\sigma_{B_k^c}(\omega).$$

Thus, for all $x\in E$

$$\begin{aligned}\mathbb{P}_x\big(\lim_{k\to\infty}\sigma_{B_k^c}<\infty\big) &\leq \mathbb{P}_x\big(\lim_{t\to 0}\lim_{k\to\infty}\sigma_{B_k^c}\circ\theta_t<\infty\big)\leq\liminf_{t\to 0}\mathbb{P}_x\big(\lim_{k\to\infty}\sigma_{B_k^c}\circ\theta_t<\infty)\big)\\ &= \liminf_{t\to 0}\mathbb{E}_x\big[\mathbb{P}_{X_t}\big(\lim_{k\to\infty}\sigma_{B_k^c}<\infty)\big]=0.\end{aligned}$$

The last equality holds true since $\mathbb{E}_x\big[\mathbb{P}_{X_t}(\lim_{k\to\infty}\sigma_{B_k^c}<\infty)\big]=\mathbb{E}_x\big[\mathbb{P}_{X_t}(\lim_{k\to\infty}\sigma_{B_k^c}<\infty)\,;\,X_t\notin N\big]=0$ for all $x\in E$. □

Lemma 3.4.11. *$\ell_t^k=\ell_t^{k+1}$, $\forall t<\sigma_{B_k^c}$ $\mathbb{P}_x$-a.s. for all $x\in B_k$ where ℓ_t^k is the positive continuous additive functional of X^{B_k} in the strict sense associated to $1_{B_k}\cdot\frac{\rho\sigma}{2}\in S_{00}^{B_k}$. In particular $\ell_t:=\lim_{k\to\infty}\ell_t^k$, $t\geq 0$, is well defined in $A_{c,1}^{+,E}$, and related to $\frac{\rho\sigma}{2}$ via the Revuz correspondence.*

Proof. Fix $f\in\mathcal{B}_b^+(B_k)$ and for $x\in B_{k+1}$ define

$$f_k(x):=\mathbb{E}_x\Big[\int_0^{\sigma_{B_k^c}}e^{-t}f(X_t)\,d\ell_t^{k+1}\Big].$$

Note that

$$\begin{aligned} f_k(x) &= \mathbb{E}_x\Big[\int_0^{\sigma_{B^c_{k+1}}} e^{-t}\, f(X_t)\, d\ell^{k+1}_t\Big] - \mathbb{E}_x\Big[\int_{\sigma_{B^c_k}}^{\sigma_{B^c_{k+1}}} e^{-t}\, f(X_t)\, d\ell^{k+1}_t\Big] \\ &= R_1^{B_{k+1}}\Big(f\ 1_{B_{k+1}}\cdot\frac{\rho\sigma}{2}\Big)(x) - \mathbb{E}_x\Big[\int_{\sigma_{B^c_k}}^{\sigma_{B^c_{k+1}}} e^{-t}\, f(X_t)\, d\ell^{k+1}_t\Big] \\ &= R_1^{B_{k+1}}\Big(f\ 1_{B_{k+1}}\cdot\frac{\rho\sigma}{2}\Big)(x) - \Big(R_1^{B_{k+1}}\Big(f 1_{B_{k+1}}\cdot\frac{\rho\sigma}{2}\Big)\Big)_{B^c_k}(x), \end{aligned}$$

where $\Big(R_1^{B_{k+1}}\Big(f1_{B_{k+1}}\cdot\frac{\rho\sigma}{2}\Big)\Big)_{B^c_k}$ is a reduced function of $R_1^{B_{k+1}}\Big(f1_{B_{k+1}}\cdot\frac{\rho\sigma}{2}\Big)$. Since $f_k\in D(\mathcal{E}^{B_{k+1}})$ and $f_k=0$ $\mathcal{E}$-q.e. on B^c_k, we have $f_k\in D(\mathcal{E}^{B_k})$. For $x\in B_k$

$$R_1^{B_k}\Big(f\ 1_{B_k}\cdot\frac{\rho\sigma}{2}\Big)(x) = \mathbb{E}_x\Big[\int_0^{\sigma_{B^c_k}} e^{-t}\, f(X_t)\, d\ell^k_t\Big].$$

Then, for $g\in\mathcal{B}^+_b(B_k)\cap L^2(B_k,m)$

$$\begin{aligned} \mathcal{E}_1^{B_k}\Big(f_k,\, R_1^{B_k}g\Big) &= \mathcal{E}_1^{B_{k+1}}\Big(f_k,\, R_1^{B_k}g\Big) \\ &= \mathcal{E}_1^{B_{k+1}}\Big(R_1^{B_{k+1}}\Big(f\ 1_{B_{k+1}}\cdot\frac{\rho\sigma}{2}\Big) - \Big(R_1^{B_{k+1}}\Big(f\ 1_{B_{k+1}}\cdot\frac{\rho\sigma}{2}\Big)\Big)_{B^c_k},\, R_1^{B_k}g\Big) \\ &= \mathcal{E}_1^{B_{k+1}}\Big(R_1^{B_{k+1}}\Big(f\ 1_{B_{k+1}}\cdot\frac{\rho\sigma}{2}\Big),\, R_1^{B_k}g\Big) \\ &= \int_{\partial G} R_1^{B_k}g\ f\, 1_{B_{k+1}}\cdot\frac{\rho d\sigma}{2} = \int_{\partial G} R_1^{B_k}g\ f\, 1_{B_k}\cdot\frac{\rho d\sigma}{2} \\ &= \mathcal{E}_1^{B_k}\Big(R_1^{B_k}\Big(f\, 1_{B_k}\cdot\frac{\rho\sigma}{2}\Big),\, R_1^{B_k}g\Big). \end{aligned}$$

Therefore, $f_k = R_1^{B_k}\Big(f\, 1_{B_k}\cdot\frac{\rho\sigma}{2}\Big)$ m-a.e. Since $R_1^{B_k}\Big(f\, 1_{B_k}\cdot\frac{\rho\sigma}{2}\Big)$ is 1-excessive for $(R^{B_k}_\alpha)_{\alpha>0}$, we obtain for any $x\in B_k$

$$\begin{aligned} R_1^{B_k}\Big(f\ 1_{B_k}\cdot\frac{\rho\sigma}{2}\Big)(x) &= \lim_{\alpha\to\infty}\alpha R^{B_k}_{\alpha+1}\Big(R_1^{B_k}\Big(f\ 1_{B_k}\cdot\frac{\rho\sigma}{2}\Big)\Big)(x) \\ &= \lim_{\alpha\to\infty}\alpha\int_{B_k} r^{B_k}_{\alpha+1}(x,y)\ R_1^{B_k}\Big(f\, 1_{B_k}\cdot\frac{\rho\sigma}{2}\Big)(y)\ m(dy) \\ &= \lim_{\alpha\to\infty}\alpha\int_{B_k} r^{B_k}_{\alpha+1}(x,y)\ f_k(y)\ m(dy) = \lim_{\alpha\to\infty}\alpha R^{B_k}_{\alpha+1} f_k(x). \end{aligned}$$

Using in particular the strong Markov property, we obtain by direct calculation that the right hand limit equals $f_k(x)$ for any $x \in B_k$. Thus, we showed for all $x \in B_k$

$$\mathbb{E}_x\left[\int_0^{\sigma_{B_k^c}} e^{-t}\ f(X_t)\ d\ell_t^k\right] = \mathbb{E}_x\left[\int_0^{\sigma_{B_k^c}} e^{-t}\ f(X_t)\ d\ell_t^{k+1}\right].$$

This implies that $\ell_t^k = \ell_t^{k+1}$, $\forall t < \sigma_{B_k^c}$ $\mathbb{P}_x$-a.s. for all $x \in B_k$ (see e.g. [15, IV. (2.12) Proposition]). □

Theorem 3.4.12. *The process* $\mathbb{M}$ *satisfies*

$$X_t = x + W_t + \int_0^t \frac{\nabla\rho}{2\,\rho}(X_s)\,ds + \int_0^t \eta(X_s)\,d\ell_s\ ,\quad t \geq 0$$

$\mathbb{P}_x$*-a.s. for all* $x \in E$ *where* W *is a standard d-dimensional Brownian motion starting from zero and* ℓ *is the positive continuous additive functional of* X *in the strict sense associated via the Revuz correspondence (cf. [35, Theorem 5.1.3]) with the weighted surface measure* $\frac{1}{2}\rho\sigma$ *on* $E \cap \partial G$.

Proof. Let $k \to \infty$ in (3.50). Then the statement follows immediately from Lemmas 3.4.10 and 3.4.11. □

Chapter 4

Non-symmetric distorted Brownian motion

4.1 Elliptic regularity and construction of a diffusion process

We shall assume (HS1)-(HS3) below throughout up to including Section 4.3:

(HS1) $\rho = \xi^2$, $\xi \in H^{1,2}_{loc}(\mathbb{R}^d, dx)$, $\rho > 0$ dx-a.e. and

$$\frac{\|\nabla \rho\|}{\rho} \in L^p_{loc}(\mathbb{R}^d, m), \quad m := \rho dx,$$

$p := (d + \varepsilon) \vee 2$ for some $\varepsilon > 0$.

By (HS1) the symmetric positive definite bilinear form

$$\mathcal{E}^0(f,g) := \frac{1}{2}\int_{\mathbb{R}^d} \langle \nabla f, \nabla g \rangle \; dm, \quad f,g \in C_0^\infty(\mathbb{R}^d)$$

is closable in $L^2(\mathbb{R}^d, m)$ and its closure $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a symmetric, strongly local, regular Dirichlet form. We further assume

(HS2) $B : \mathbb{R}^d \to \mathbb{R}^d$, $\|B\| \in L^p_{loc}(\mathbb{R}^d, m)$ where p is the same as in (HS1) and

$$\int_{\mathbb{R}^d} \langle B, \nabla f \rangle \; dm = 0, \quad \forall f \in C_0^\infty(\mathbb{R}^d), \tag{4.1}$$

and

(HS3)
$$\left|\int_{\mathbb{R}^d} \langle B, \nabla f\rangle \; g \; \rho \; dx\right| \le c_0 \; \mathcal{E}_1^0(f,f)^{1/2} \; \mathcal{E}_1^0(g,g)^{1/2}, \quad \forall f, g \in C_0^\infty(\mathbb{R}^d),$$

where c_0 is some constant (independent of f and g) and $\mathcal{E}_\alpha^0(\cdot,\cdot) := \mathcal{E}^0(\cdot,\cdot) + \alpha(\cdot,\cdot)_{L^2(\mathbb{R}^d,m)}$, $\alpha > 0$.

Next, we consider the non-symmetric bilinear form

$$\mathcal{E}(f,g) := \frac{1}{2}\int_{\mathbb{R}^d} \langle \nabla f, \nabla g\rangle \; dm - \int_{\mathbb{R}^d} \langle B, \nabla f\rangle \; g \; dm, \quad f, g \in C_0^\infty(\mathbb{R}^d) \tag{4.2}$$

in $L^2(\mathbb{R}^d, m)$. Then by (HS1)-(HS3) $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ is closable in $L^2(\mathbb{R}^d, m)$ and the closure $(\mathcal{E}, D(\mathcal{E}))$ is a non-symmetric Dirichlet form (cf. [44, II. 2. d)]). Let $(T_t)_{t>0}$ (resp. $(\hat{T}_t)_{t>0}$) and $(G_\alpha)_{\alpha>0}$ (resp. $(\hat{G}_\alpha)_{\alpha>0}$) be the $L^2(\mathbb{R}^d, m)$-semigroup (resp. cosemigroup) and resolvent (resp. coresolvent) associated to $(\mathcal{E}, D(\mathcal{E}))$ and $(L, D(L))$ (resp. $(\hat{L}, D(\hat{L}))$) be the corresponding generator (resp. cogenerator) (see [44, Diagram 3, p. 39]). Using properties (HS2) and [44, I Proposition 4.7] (cf. also [44, II 2. d)]), it is straightforward to see that $(T_t)_{t>0}$ as well as $(\hat{T}_t)_{t>0}$ are submarkovian. Here an operator S is called submarkovian if $0 \le f \le 1$ implies $0 \le Sf \le 1$. It is then further easy to see that $(T_t)_{t>0}$ (resp. $(G_\lambda)_{\lambda>0}$) restricted to $L^r(\mathbb{R}^d, m) \cap L^\infty(\mathbb{R}^d, m)$ can be extended to strongly continuous contraction semigroups (resp. strongly continuous contraction resolvents) on all $L^r(\mathbb{R}^d, m)$, $r \in [1, \infty)$ (see [44, I. 1] for the definition of a strongly continuous contraction semigroup (resp. resolvent)).

We denote the corresponding operator families again by $(T_t)_{t>0}$ and $(G_\lambda)_{\lambda>0}$ and let $(L_r, D(L_r))$ be the corresponding generator on $L^r(\mathbb{R}^d, m)$. Since by (HS1), (HS2), $\left\|\frac{\nabla\rho}{2\rho}\right\|$, $\|B\| \in L^p_{loc}(\mathbb{R}^d, m)$, we get $C_0^\infty(\mathbb{R}^d) \subset D(L_r)$ for any $r \in [1, p]$ and

$$L_r u = \frac{1}{2}\Delta u + \langle \frac{\nabla \rho}{2\rho} + B, \nabla u\rangle, \quad u \in C_0^\infty(\mathbb{R}^d), \quad r \in [1, p]. \tag{4.3}$$

Let us first state an elliptic regularity result (cf. [16, Theorem 1 (iii)(b)], [17, Remark 2.15]). Its consequences in the symmetric case were discussed in [6]. Likewise the Corollaries 4.1.3, 4.1.4, 4.1.6, and Remark 4.1.5, 4.1.7 below can be obtained.

Proposition 4.1.1. *Let E be an open set in $\mathbb{R}^d$ and $A : E \to \mathbb{R}^d$, $c : E \to \mathbb{R}$ Borel measurable maps. Suppose μ is a (signed) Radon measure on E and $f \in L^1_{loc}(E, dx)$*

such that $\|A\|,\ c \in L^1_{loc}(E,\mu)$ *and*

$$\int Nu(x)\ \mu(dx) = \int u(x)\ f(x)\ dx, \quad \forall u \in C_0^\infty(E),$$

where

$$Nu(x) := \Delta u(x) + \langle A(x), \nabla u(x)\rangle + c(x)\ u(x).$$

If for some $\tilde{p} > d$, $\|A\| \in L^{\tilde{p}}_{loc}(E,\mu)$, $c \in L^{\tilde{p}d/(\tilde{p}+d)}_{loc}(E,\mu)$, *and* $f \in L^{\tilde{p}d/(\tilde{p}+d)}_{loc}(E,dx)$, *then* $\mu = \rho dx$ *with* ρ *continuous and*

$$\rho \in H^{1,\tilde{p}}_{loc}(E,dx)\ \Big(\subset C^{1-d/\tilde{p}}_{loc}(E)\Big),$$

where $C^{1-d/\tilde{p}}_{loc}(E)$ *denotes the set of all locally Hölder continuous functions of order* $1-d/\tilde{p}$ *on* E. *If* $E_0 := E \cap \{\rho > 0\}$ *and moreover* $f, c \in L^{\tilde{p}}_{loc}(E_0)$, *then for any open ball* $B \subset \overline{B} \subset E_0$ *there exists* $c_B \in (0,\infty)$ *(independent of* ρ *and* f*) such that*

$$\|\rho\|_{H^{1,\tilde{p}}(B,dx)} \le c_B \Big(\|\rho\|_{L^1(B,dx)} + \|f\|_{L^{\tilde{p}}(B,dx)}\Big).$$

Remark 4.1.2. *At first side the assumption that the drift in* (1.7) *or the first order coefficient in* (4.2) *is of type* $b := \frac{\nabla\rho}{2\rho} + B$ *looks rather special. But the* $L^p_{loc}(\mathbb{R}^d, m)$ *condition makes it very natural, because the special form of* b *follows, if one considers the operator*

$$Lu := \Delta u + \langle b, \nabla u\rangle\ , \quad u \in C_0^\infty(\mathbb{R}^d)\ ,$$

and assumes that if has an infinitesimally (not necessarily probability) invariant measure m, *i.e.* m *is a nonnegative Radon measure* m *on* $\mathbb{R}^d$, *such that* $b \in L^p_{loc}(\mathbb{R}^d, m)$ *and*

$$\int Lu\ \mathrm{d}m = 0 \quad \forall u \in C_0^\infty(\mathbb{R}^d)\ .$$

Then it follows by Proposition 4.1.1 that $m = \rho\mathrm{d}x$ *and that* ρ *satisfies (HS1). Defining*

$$B := b - \frac{\nabla\rho}{2\rho}\ ,$$

it satisfies (HS2). So, we have the above decomposition in a natural way.

Corollary 4.1.3. ρ *is in* $H^{1,p}_{loc}(\mathbb{R}^d, dx)$ *and* ρ *has a continuous* dx*-version in* $C^{1-d/p}_{loc}(\mathbb{R}^d)$.

Proof. By (4.1), (4.3) and integration by parts, we obtain

$$\int Lu \,\mathrm{d}m = 0, \quad \forall u \in C_0^\infty(\mathbb{R}^d).$$

Since $\frac{\|\nabla\rho\|}{\rho}, \|B\| \in L^p_{loc}(\mathbb{R}^d, m)$, the assertion follows by Proposition 4.1.1 applied with $\tilde{p} = p$. □

From now on, we shall always consider the continuous dx-version of ρ and denote it also by ρ.

Corollary 4.1.4. *Let $\lambda > 0$. Suppose $g \in L^r(\mathbb{R}^d, m)$, $r \in [p, \infty)$. Then*

$$\rho\ G_\lambda g \in H^{1,p}_{loc}(\mathbb{R}^d, dx)$$

and for any open ball $B \subset \overline{B} \subset \{\rho > 0\}$ there exists $c_{B,\lambda} \in (0, \infty)$, independent of g, such that

$$\|\ \rho\ G_\lambda g\ \|_{H^{1,p}(B,dx)} \leq c_{B,\lambda}\ \Big(\|G_\lambda g\|_{L^1(B,dm)} + \|g\|_{L^p(B,dm)}\Big). \tag{4.4}$$

Proof. Let $g \in C_0^\infty(\mathbb{R}^d)$. Then we have

$$\int (\lambda - \hat{L})u\ G_\lambda g\ \rho\ dx = \int u\ g\ \rho\ dx, \quad \forall u \in C_0^\infty(\mathbb{R}^d),$$

where

$$\hat{L}u = \frac{1}{2}\Delta u + \langle \frac{\nabla\rho}{2\rho} - B, \nabla u\rangle.$$

Now we apply Proposition 4.1.1 with $\mu = -\frac{1}{2}\rho G_\lambda g dx$ and $N = -2(\lambda - \hat{L})$ and $f = g\rho$ to prove the assertion for $g \in C_0^\infty(\mathbb{R}^d)$. Since $C_0^\infty(\mathbb{R}^d)$ is dense in $(L^r(\mathbb{R}^d, m), \|\cdot\|_{L^r(\mathbb{R}^d,m)})$, $r \in [1, \infty)$, the assertion for general $g \in L^r(\mathbb{R}^d, m)$ follows by continuity and (4.4). □

Remark 4.1.5. *By [44, I. Corollary 2.21], it holds that $(T_t)_{t>0}$ is analytic on $L^2(\mathbb{R}^d, m)$. By Stein interpolation (cf. e.g. [8, Lecture 10, Theorem 10.8]) $(T_t)_{t>0}$ is also analytic on $L^r(\mathbb{R}^d, m)$ for all $r \in (2, \infty)$. We would like to thank Hendrik Vogt for pointing this out to us as well as a misprint in the mentioned Theorem 10.8. There θ_τ should be defined as $\tau \cdot \theta$ and not as $(1 - \tau) \cdot \theta$.*

Corollary 4.1.6. *Let $t > 0$, $r \in [p, \infty)$.*

(i) *Let* $u \in D(L_r)$. *Then*

$$\rho\ T_t u \in H^{1,p}_{loc}(\mathbb{R}^d, dx)$$

and for any open ball $B \subset \overline{B} \subset \{\rho > 0\}$ *there exists* $c_B \in (0,\infty)$ *(independent of* u *and* t*) such that*

$$\begin{aligned} & \|\rho\ T_t u\|_{H^{1,p}(B,dx)} \\ \le\ & c_B \left(\|T_t u\|_{L^1(B,m)} + \|T_t(1-L_r)u\|_{L^p(\mathbb{R}^d,m)} \right) \\ \le\ & c_B \left(m(B)^{\frac{r-1}{r}} \|u\|_{L^r(\mathbb{R}^d,m)} + m(B)^{\frac{r-p}{rp}} \|(1-L_r)u\|_{L^r(\mathbb{R}^d,m)} \right). \end{aligned} \tag{4.5}$$

(ii) *Let* $f \in L^r(\mathbb{R}^d, m)$. *Then the above statements still hold with* (4.5) *replaced by*

$$\|\rho\ T_t f\|_{H^{1,p}(B,dx)} \le \tilde{c}_B\ t^{-1} \|f\|_{L^r(\mathbb{R}^d,m)},$$

where $\tilde{c}_B \in (0,\infty)$ *(independent of* f, t*).*

Remark 4.1.7. *By* (4.5) *and Sobolev imbedding, for* $r \in [p,\infty)$, $R > 0$ *the set*

$$\{T_t u \mid t > 0,\ u \in D(L_r),\ \|u\|_{L^r(\mathbb{R}^d,m)} + \|L_r u\|_{L^r(\mathbb{R}^d,m)} \le R\}$$

is equicontinuous on $\{\rho > 0\}$.

From now on, we shall keep the notation

$$E := \{\rho > 0\}.$$

By Corollaries 4.1.3, 4.1.4, 4.1.6 and Remark 4.1.5, 4.1.7, exactly as in [6, Section 3], we obtain the existence of a transition kernel density $p_t(\cdot,\cdot)$ on the open set E such that

$$P_t f(x) := \int_E f(y) p_t(x,y)\ m(dy), \quad x \in E,\ t > 0$$

is a (temporally homogeneous) submarkovian transition function (cf. [22, 1.2]) and an m-version of $T_t f$ for any $f \in \cup_{r \ge p} L^r(E,m)$. Moreover, letting $P_0 := id$, it holds

$$P_t f \in C(E) \quad \forall f \in \cup_{r \ge p} L^r(E,m) \tag{4.6}$$

and

$$\lim_{t \to 0} P_{t+s} f(x) = P_s f(x) \quad \forall s \ge 0,\ x \in E,\ f \in C_0^\infty(\mathbb{R}^d). \tag{4.7}$$

By a 3ε-argument (4.7) extends to $C_0(\mathbb{R}^d)$. Similarly, since for $\lambda > 0$, $f \in L^p(E,m)$, $G_\lambda f$ has a unique continuous m-version on E by Corollary 4.1.4 as in [6, Lemma 3.4, Proposition 3.5], we can find $(R_\lambda)_{\lambda>0}$ with resolvent kernel density $r_\lambda(\cdot,\cdot)$ defined on $E \times E$ such that

$$R_\lambda f(x) := \int f(y)\, r_\lambda(x,y)\; m(dy), \quad x \in E,\ \lambda > 0,$$

satisfies

$$R_\lambda f \in C(E) \text{ and } R_\lambda f = G_\lambda f \;\; m\text{-a.e for any } f \in L^p(E,m). \tag{4.8}$$

We further consider

(HS4) $(\mathcal{E}, D(\mathcal{E}))$ is conservative.

Remark 4.1.8. *Consider the C_0-semigroups $(T_t)_{t>0}$, $(\hat{T}_t)_{t>0}$ of submarkovian contractions on $L^1(\mathbb{R}^d,m)$. In particular $(T_t)_{t>0}$ (and also $(\hat{T}_t)_{t>0}$) can be defined as semigroups on $L^\infty(\mathbb{R}^d,m)$. Then $(\mathcal{E}, D(\mathcal{E}))$ is called conservative, if*

$$T_t 1 = 1 \; m\text{-a.e. for some (and hence all) } t > 0 \tag{4.9}$$

Obviously, (4.9) *holds e.g. if $m(\mathbb{R}^d) < \infty$ and $\|B\| \in L^1(\mathbb{R}^d,m)$. In Section 4.4 below we shall present a whole class of examples which do not satisfy these two assumptions, but for which* (4.9)*, i.e. (HS4) holds. Clearly* (4.9) *holds, if and only if m is $(\hat{T}_t)$-invariant, that is*

$$\int \hat{T}_t f \; dm = \int f \; dm \quad \forall f \in L^1(\mathbb{R}^d,m) \tag{4.10}$$

and by [60, Corollary 2.2] (4.10) *is equivalent to*

$$(1-\hat{L})\big(C_0^\infty(\mathbb{R}^d)\big) \subset L^1(\mathbb{R}^d,m) \text{ densely.} \tag{4.11}$$

Thus (4.11) *is equivalent to (HS4).*

Following [6, Proposition 3.8], we obtain:

Proposition 4.1.9. *If (HS4) holds (additionally to (HS1)-(HS3)), then:*

(i) $\lambda R_\lambda 1(x) = 1$ *for all* $x \in E$, $\lambda > 0$.

(ii) $(P_t)_{t>0}$ *is strong Feller on E, i.e.* $P_t(\mathcal{B}_b(\mathbb{R}^d)) \subset C_b(E)$ *for all* $t > 0$.

(iii) $P_t 1(x) = 1$ *for all* $x \in E$, $t > 0$.

By [44, V. 2.12 (ii)] (see also [71, Proposition 1]), it follows that $(\mathcal{E}, D(\mathcal{E}))$ is strictly quasi-regular. Actually, in [71, Section 4.1], it is shown that this is even true for non-sectorial B, i.e. when $\|B\|$ is merely in $L^2_{loc}(\mathbb{R}^d, m)$. In particular, by [44, V.2.13] (see also [71, Theorem 3] for the non-sectorial case) there exists a Hunt process $\tilde{\mathbb{M}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}})_{t\geq 0}, (\tilde{X}_t)_{t\geq 0}, (\tilde{\mathbb{P}}_x)_{x\in\mathbb{R}^d\cup\{\Delta\}})$ with the lifetime $\zeta := \inf\{t \geq 0 \,|\, \tilde{X}_t = \Delta\}$ and the cemetery Δ such that $(\mathcal{E}, D(\mathcal{E}))$ is (strictly properly) associated with $\tilde{\mathbb{M}}$.
Consider the strict capacity $\mathrm{Cap}_{\mathcal{E}}$ of the non-symmetric Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ as defined in [44, V.2.1] and [71, Definition 1], i.e.

$$\mathrm{Cap}_{\mathcal{E}} = \mathrm{cap}_{1,\hat{G}_1\varphi}$$

for some fixed $\varphi \in L^1(\mathbb{R}^d, m) \cap \mathcal{B}_b(\mathbb{R}^d)$, $0 < \varphi \leq 1$. Due to the properties of smooth measures w.r.t. $\mathrm{Cap}_{\mathcal{E}}$ in [71, Section 3] it is possible to consider the work [69] with cap_φ (as defined in [69]) replaced by $\mathrm{Cap}_{\mathcal{E}}$. In particular [69, Theorem 3.10 and Proposition 4.2] apply w.r.t. the strict capacity $\mathrm{Cap}_{\mathcal{E}}$ and therefore the paths of $\tilde{\mathbb{M}}$ are continuous $\tilde{\mathbb{P}}_x$-a.s. for strictly $\mathcal{E}$-q.e. $x \in \mathbb{R}^d$ on the one-point-compactification $\mathbb{R}^d_\Delta$ of $\mathbb{R}^d$ with Δ as point at infinity. We may hence assume that

$$\tilde{\Omega} = \{\omega = (\omega(t))_{t\geq 0} \in C([0,\infty), \mathbb{R}^d_\Delta) \mid \omega(t) = \Delta \quad \forall t \geq \zeta(\omega)\} \tag{4.12}$$

and

$$\tilde{X}_t(\omega) = \omega(t), \quad t \geq 0.$$

Let Cap be the capacity related to the symmetric Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$ as defined in [35, Section 2.1]. Then, it holds $\mathrm{Cap}(\{\rho = 0\}) = 0$ by [31, Theorem 2].

Lemma 4.1.10. *Let $N \subset \mathbb{R}^d$. Then*

$$\mathrm{Cap}(N) = 0 \Rightarrow \mathrm{Cap}_{\mathcal{E}}(N) = 0.$$

In particular $\mathrm{Cap}_{\mathcal{E}}(\{\rho = 0\}) = 0$.

Proof. Let $N \subset \mathbb{R}^d$ be such that $\mathrm{Cap}(N) = 0$. Then by the definition of Cap there exist closed sets $F_k \subset \mathbb{R}^d \setminus N$, $k \geq 1$ such that

$$\lim_{k\to\infty} \mathrm{Cap}(\mathbb{R}^d \setminus F_k) = 0.$$

Therefore, we may assume that $\mathrm{Cap}(\mathbb{R}^d \setminus F_k) < \infty$ for any $k \geq 1$. Hence

$$\mathcal{L}_{\mathbb{R}^d\setminus F_k} := \{u \in D(\mathcal{E}^0) \mid u \geq 1 \ m\text{-a.e. on } \mathbb{R}^d \setminus F_k\} \neq \emptyset, \quad \forall k \geq 1.$$

Then by [35, Lemma 2.1.1.] there exists a unique element $e_{\mathbb{R}^d\setminus F_k} \in \mathcal{L}_{\mathbb{R}^d\setminus F_k}$ such that

$$\mathrm{Cap}(\mathbb{R}^d \setminus F_k) = \mathcal{E}^0(e_{\mathbb{R}^d\setminus F_k}, e_{\mathbb{R}^d\setminus F_k}) \ \text{ and } \ e_{\mathbb{R}^d\setminus F_k} = 1 \ m\text{-a.e on } \ \mathbb{R}^d \setminus F_k.$$

We denote by $\mathcal{P}$ the family of 1-excessive functions w.r.t. $\mathcal{E}$ in $D(\mathcal{E})$ and denote by h_U the (1-) reduced function on an open set $U \subset \mathbb{R}^d$ of a function h in $D(\mathcal{E})$. Then by (HS3) and [44, III. Proposition 1.5] for $u \leq 1, \ u \in \mathcal{P}$

$$\mathcal{E}_1(u_{\mathbb{R}^d\setminus F_k}, u_{\mathbb{R}^d\setminus F_k}) \leq \mathcal{E}_1(u_{\mathbb{R}^d\setminus F_k}, e_{\mathbb{R}^d\setminus F_k}) \leq K\, \mathcal{E}_1(u_{\mathbb{R}^d\setminus F_k}, u_{\mathbb{R}^d\setminus F_k})^{1/2}\mathcal{E}_1(e_{\mathbb{R}^d\setminus F_k}, e_{\mathbb{R}^d\setminus F_k})^{1/2},$$

where K is the sector constant. Therefore,

$$\lim_{k\to\infty} \sup_{\substack{u\leq 1,\\ u\in\mathcal{P}}} \mathcal{E}_1(u_{\mathbb{R}^d\setminus F_k}, u_{\mathbb{R}^d\setminus F_k}) = 0.$$

Since for any fixed $\varphi \in L^1(\mathbb{R}^d, m) \cap \mathcal{B}_b(\mathbb{R}^d)$, $0 < \varphi \leq 1$

$$\mathcal{E}_1(u_{\mathbb{R}^d\setminus F_k}, \hat{G}_1\varphi) \leq K \ \mathcal{E}_1(u_{\mathbb{R}^d\setminus F_k}, u_{\mathbb{R}^d\setminus F_k})^{1/2}\mathcal{E}_1(\hat{G}_1\varphi, \hat{G}_1\varphi)^{1/2},$$

we have

$$\mathrm{Cap}_{\mathcal{E}}(N) \leq \lim_{k\to\infty} \sup_{\substack{u\leq 1,\\ u\in\mathcal{P}}} \mathcal{E}_1(u_{\mathbb{R}^d\setminus F_k}, \hat{G}_1\varphi) = 0.$$

□

For a Borel set $B \subset \mathbb{R}^d$, we define

$$\sigma_B := \inf\{t > 0 \mid X_t \in B\}, \quad D_B := \inf\{t \geq 0 \mid X_t \in B\}.$$

Let

$$\tilde{X}_t^E(\omega) := \begin{cases} \tilde{X}_t(\omega) & 0 \leq t < D_{\mathbb{R}^d\setminus E}(\omega) \\ \Delta & t \geq D_{\mathbb{R}^d\setminus E}(\omega), \ \omega \in \tilde{\Omega}. \end{cases}$$

Then $\tilde{\mathbb{M}}^E := (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, (\tilde{X}_t^E)_{t\geq 0}, (\tilde{\mathbb{P}}_x)_{x\in E\cup\{\Delta\}})$ is again a Hunt Process by [35, Theorem A.2.10] and its lifetime is $\zeta^E := \zeta \wedge D_{\mathbb{R}^d\setminus E}$. $\tilde{\mathbb{M}}^E$ is called the part process of $\tilde{\mathbb{M}}$ on E and it is associated with the part $(\mathcal{E}^E, D(\mathcal{E}^E))$ of $(\mathcal{E}, D(\mathcal{E}))$ on E (cf. [48, Theorem 3.5.7]). We denote the $L^2(E,m)$-semigroup of $(\mathcal{E}^E, D(\mathcal{E}^E))$ by $(T_t^E)_{t>0}$.

Lemma 4.1.11. *Let $(F_k)_{k\geq 1}$ be an increasing sequence of compact subsets of E with $\cup_{k\geq 1} F_k = E$ and such that $F_k \subset \mathring{F}_{k+1}$, $k \geq 1$(here $\mathring{F}$ denotes the interior of F). Then*

$$\tilde{\mathbb{P}}_x(\tilde{\Omega}_0) = 1 \text{ for strictly } \mathcal{E}\text{-q.e. } x \in E,$$

where

$$\tilde{\Omega}_0 := \tilde{\Omega} \cap \{\omega \mid \omega(0) \in E \cup \{\Delta\} \text{ and } \lim_{k\to\infty} \sigma_{E\setminus F_k}(\omega) \geq \zeta(\omega)\}.$$

Proof. First note that $\tilde{\mathbb{P}}_x(\zeta = \zeta^E) = 1$ for m-a.e. $x \in E$ since $\mathrm{Cap}(\mathbb{R}^d \setminus E) = 0$. By [44, IV. Theorem 5.1 and Proposition 5.30] there exists an increasing sequence of compact subsets $(K)_{n\geq 1}$ of E such that

$$\tilde{\mathbb{P}}_x(\lim_{n\to\infty} \sigma_{E\setminus K_n} \geq \zeta^E) = 1 \text{ for } m\text{-a.e. } x \in E.$$

The last and previous imply that

$$\tilde{\mathbb{P}}_x(\lim_{k\to\infty} \sigma_{E\setminus F_k} \geq \zeta) = 1 \text{ for } m\text{-a.e. } x \in E \tag{4.13}$$

since $(\mathring{F}_k)_{k\geq 1}$ is an open cover of K_n for every $n \geq 1$. (4.12) and (4.13) now easily imply the assertion.

□

Theorem 4.1.12. *There exists a Hunt process*

$$\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E_\Delta})$$

with state space E, having the transition function $(P_t)_{t\geq 0}$ as transition semigroup. In particular $\mathbb{M}$ satisfies the absolute continuity condition, because

$$T_t^E f = P_t f \quad m\text{-a.e. } \forall t > 0,\ f \in L^2(E,m) \cap \mathcal{B}_b(E).$$

Moreover $\mathbb{M}$ has continuous sample paths in the one point compactification E_Δ of E with the cemetery Δ as point at infinity.

Proof. Given the transition function $(P_t)_{t\geq 0}$ we can construct $\mathbb{M}$ with continuous sample paths in E_Δ following the line of arguments in [6] (see also Section 3.1.1 using in particular Lemma 4.1.11 and our further previous preparations. As in Lemma 3.3.2, we then show that the (temporally homogeneous) sub-Markovian transition function $(P_t)_{t\geq 0}$ on $\big(E, \mathcal{B}(E)\big)$ with transition kernel density $p_t(\cdot,\cdot)$ on $E\times E$ satisfies

$$T_t^E f = T_t f = P_t f \quad m\text{-a.e.}$$

for any $t>0$ and $f\in\mathcal{B}_b(E)$ with compact support (i.e. $|f|dm$ has compact support). Thus the absolute continuity condition is satisfied. □

Remark 4.1.13. *If in addition (HS4) holds, one can drop Δ in Theorem 4.1.12 and $\mathbb{M}$ becomes a classical (conservative) diffusion with state space E. Indeed, it then holds*

$$\mathbb{P}_x(\zeta=\infty)=1, \quad \forall x\in E.$$

4.2 Existence of weak solutions

Lemma 4.2.1. *Assume (HS1)-(HS3).*

(i) Let $f\in\bigcup_{s\in[p,\infty)}L^s(m)$, $f\geq 0$, then for all $t>0$, $x\in E$,

$$\int_0^t P_s f(x)\ ds<\infty,$$

hence

$$\int\int_0^t f(X_s)\ ds\ d\mathbb{P}_x<\infty.$$

(ii) Let $u\in C_0^\infty(\mathbb{R}^d)$, $\lambda>0$. Then

$$R_\lambda\big((\lambda-L)u\big)(x)=u(x) \quad \forall x\in E.$$

(iii) Let $u\in C_0^\infty(\mathbb{R}^d)$, $t>0$. Then

$$P_t u(x)-u(x)=\int_0^t P_s(Lu)(x)\ ds \quad \forall x\in E.$$

Proof. The proof is the same as the one for [6, Lemma 5.1]. □

Lemma 4.2.2. *For* $u \in C_0^\infty(\mathbb{R}^d)$

$$Lu^2 - 2u\ Lu = \|\nabla u\|^2.$$

Proof. Using (4.3) we obtain for $u \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned} Lu^2 - 2u\ Lu &= \|\nabla u\|^2 + u\Delta u + \langle \frac{\nabla \rho}{2\rho} + B, 2u\nabla u\rangle \\ &\quad -u\Delta u - 2u\ \langle \frac{\nabla \rho}{2\rho} + B, \nabla u\rangle = \|\nabla u\|^2. \end{aligned}$$

□

Theorem 4.2.3. *Let* $u \in C_0^\infty(\mathbb{R}^d)$ *and*

$$M_t := \left(u(X_t) - u(X_0) - \int_0^t Lu(X_r)\ dr \right)^2 - \int_0^t \|\nabla u\|^2(X_r)\ dr, \quad t \geq 0.$$

Then $(M_t)_{t\geq 0}$ *is an* $(\mathcal{F}_t)_{t\geq 0}$*-martingale under* $\mathbb{P}_x$, $\forall x \in E$.

Proof. By Lemma 4.2.1 and the Markov property

$$u(X_t) - u(X_0) - \int_0^t Lu(X_r)\ dr, \quad u \in C_0^\infty(\mathbb{R}^d),\ t \geq 0$$

is a square integrable $(\mathcal{F}_t)_{t\geq 0}$-martingale under $\mathbb{P}_x$ for all $x \in E$. Fix $x \in E$, $u \in C_0^\infty(\mathbb{R}^d)$, and set

$$M_t := \left(u(X_t) - u(X_0) - \int_0^t Lu(X_r)\ dr \right)^2 - \int_0^t \|\nabla u\|^2(X_r)\ dr, \quad t \geq 0.$$

Then since $u \in D(L_p)$ (cf. (4.3)), it follows by Lemma 4.2.1 that $(M_t)_{t\geq 0}$ and all integrands below are integrable w.r.t. $\mathbb{P}_x$. Using Lemma 4.2.2 we get for $s \in [0,t)$

$$\begin{aligned}
& M_t - M_s \\
= \; & \Big(u(X_t) - u(X_0) - \int_0^t Lu(X_r)\, dr + u(X_s) - u(X_0) - \int_0^s Lu(X_r)\, dr\Big) \\
& \times \Big(u(X_t) - u(X_s) - \int_s^t Lu(X_r)\, dr\Big) - \int_s^t (Lu^2 - 2u\; Lu)(X_r)\, dr \\
= \; & \Big(u(X_t) + u(X_s) - 2u(X_0) - 2\int_0^s Lu(X_r)\, dr - \int_s^t Lu(X_r)\, dr\Big) \\
& \times \Big(u(X_t) - u(X_s) - \int_s^t Lu(X_r)\, dr\Big) - \int_s^t (Lu^2 - 2u\; Lu)(X_r)\, dr \\
= \; & u^2(X_t) - u^2(X_s) - 2u(X_0)\Big(u(X_t) - u(X_s)\Big) \\
& -2\Big(u(X_t) - u(X_s)\Big)\int_0^s Lu(X_r)\, dr - \Big(u(X_t) - u(X_s)\Big)\int_0^{t-s} Lu(X_{r+s})\, dr \\
& -\Big(u(X_t) + u(X_s)\Big)\int_0^{t-s} Lu(X_{r+s})\, dr + 2u(X_0)\int_0^{t-s} Lu(X_{r+s})\, dr \\
& +2\int_0^s Lu(X_r)\, dr \int_0^{t-s} Lu(X_{r+s})\, dr + \Big(\int_0^{t-s} Lu(X_{r+s})\, dr\Big)^2 \\
& -\int_s^t \Big(Lu^2 - 2u\; Lu\Big)(X_r)\, dr.
\end{aligned}$$

Taking conditional expectation, it follows $\mathbb{P}_x$-a.s.

$$\begin{aligned}
& \mathbb{E}_x[M_t - M_s \mid \mathcal{F}_s] = P_{t-s}u^2(X_s) - u^2(X_s) \\
& -2u(x)\Big(P_{t-s}u(X_s) - u(X_s)\Big) - 2\Big(P_{t-s}u(X_s) - u(X_s)\Big)\int_0^s Lu(X_r)\, dr \\
& -2\mathbb{E}_x\Big[u(X_t)\int_0^{t-s} Lu(X_{r+s})\, dr \mid \mathcal{F}_s\Big] + 2u(x)\int_0^{t-s} P_r(Lu)(X_s)\, dr \\
& +2\int_0^s Lu(X_r)\, dr \int_0^{t-s} P_r(Lu)(X_s)\, dr + \mathbb{E}_x\Big[\Big(\int_0^{t-s} Lu(X_{r+s})\, dr\Big)^2 \mid \mathcal{F}_s\Big] \\
& -\int_0^{t-s} P_r\Big(Lu^2 - 2u\; Lu\Big)(X_s)\, dr.
\end{aligned}$$

Using Lemma 4.2.1(iii) this simplifies to

$$\mathbb{E}_x[M_t - M_s \mid \mathcal{F}_s] = -2\mathbb{E}_x\Big[u(X_t)\int_0^{t-s} Lu(X_{r+s})\ dr \mid \mathcal{F}_s\Big]$$
$$+\mathbb{E}_x\Big[\Big(\int_0^{t-s} Lu(X_{r+s})\ dr\Big)^2 \mid \mathcal{F}_s\Big] + 2\int_0^{t-s} P_r\Big(u\ Lu\Big)(X_s)\ dr.$$

Note that the first term of the right hand side satisfies

$$-2\mathbb{E}_x\Big[u(X_t)\int_0^{t-s} Lu(X_{r+s})\ dr \mid \mathcal{F}_s\Big] = -2\int_0^{t-s} P_r\Big(Lu\ P_{t-s-r}u\Big)(X_r)\ dr$$

and the second term satisfies

$$\mathbb{E}_x\Big[\Big(\int_0^{t-s} Lu(X_{r+s})\ dr\Big)^2 \mid \mathcal{F}_s\Big] = 2\int_0^{t-s}\int_0^{r'} \mathbb{E}_{X_s}\Big[Lu(X_r)Lu(X_{r'})\Big]\ dr\ dr'$$
$$= 2\int_0^{t-s}\int_0^{r'} P_r\Big(LuP_{r'-r}(Lu)\Big)(X_s)\ dr\ dr'$$
$$= 2\int_0^{t-s} P_r\Big(Lu\big(P_{t-s-r}u - u\big)\Big)(X_s)\ dr$$

by Fubini's theorem. Therefore $\mathbb{E}_x[M_t - M_s \mid \mathcal{F}_s] = 0$ $\mathbb{P}_x$-a.s. and the assertion follows. □

Let $\theta_s : \Omega \to \Omega$, $s > 0$, be the canonical shift, i.e. $\theta_s(\omega) = \omega(\cdot + s)$, $\omega \in \Omega$.

Lemma 4.2.4. *Let $(B_k)_{k\geq 1}$ be an increasing sequence of relatively compact open sets in E with $\cup_{k\geq 1} B_k = E$. Then for all $x \in E$*

$$\mathbb{P}_x\Big(\lim_{k\to\infty} \sigma_{E\setminus B_k} \geq \zeta\Big) = 1.$$

Proof. Let

$$\Lambda := \Big\{\lim_{k\to\infty} \sigma_{E\setminus B_k} \geq \zeta\Big\}.$$

Note that by Lemma 4.1.11 for m-a.e. $x \in E$

$$\mathbb{P}_x(\Lambda) = 1.$$

Then for $x \in E$ and $s > 0$

$$\begin{aligned}\mathbb{P}_x(\theta_s^{-1}(\Lambda)) &= \mathbb{E}_x[1_\Lambda \circ \theta_s] = \mathbb{E}_x\Big[\mathbb{E}_x[1_\Lambda \circ \theta_s \mid \mathcal{F}_s]\Big] = \mathbb{E}_x\Big[\mathbb{E}_{X_s}[1_\Lambda]\Big] \\ &= \int_E p_s(x,y)\ \mathbb{E}_y[1_\Lambda]\ m(dy) + (1 - P_s(x,E))\mathbb{P}_\Delta(\Lambda) = 1.\end{aligned}$$

Let $x \in E$. Define

$$\Omega_x := \{\omega \in \Omega \mid t \mapsto X_t(\omega),\ t \geq 0 \text{ is continuous in } E_\Delta \text{ and } X_0(\omega) = x\} \ \cap \bigcap_{\substack{s>0 \\ s \in S}} \theta_s^{-1} \circ \Lambda,$$

where S is a countable dense set in $(0,\infty)$. Fix $\omega \in \Omega_x$. By the continuity of $X_t(\omega)$ there is $s' \in S$ such that $X_t(\omega) \in B_{\bar{k}}$, $t \in [0, s']$, for some $\bar{k} \in \mathbb{N}$. This implies

$$\sigma_{E\setminus B_k}(\omega) = s' + \sigma_{E\setminus B_k}(\theta_{s'}(\omega))$$

for $k \geq \bar{k}$ and since $\zeta(\omega) \geq s'$, we get

$$\zeta(\omega) = s' + \zeta(\theta_{s'}(\omega)).$$

Putting all together and noting that $\theta_{s'}(\omega) \in \Lambda$, we obtain

$$\lim_{k\to\infty} \sigma_{E\setminus B_k}(\omega) = \lim_{k\to\infty} \sigma_{E\setminus B_k}(\theta_{s'}(\omega)) + s' \geq \zeta(\theta_{s'}(\omega)) + s' = \zeta(\omega).$$

Hence $\Omega_x \subset \Lambda$. Since $\mathbb{P}_x(\Omega_x) = 1$, the assertion follows. $\square$

Remark 4.2.5. *For an alternative proof of Lemma 4.2.4, which does not require the absolute continuity condition, we refer to Lemma A.0.3.*

Let

$$M_t^u := u(X_t) - u(X_0) - \int_0^t Lu(X_s)\ ds, \quad u \in C_0^\infty(E), \quad t \geq 0.$$

Clearly $(M_t^u)_{t\geq 0}$ is a continuous $(\mathcal{F}_t)_{t\geq 0}$-martingale under $\mathbb{P}_x$, $x \in E$ and for u_1, $u_2 \in C_0^\infty(E)$, $M_t^{u_1+u_2} = M_t^{u_1} + M_t^{u_2}$. By Theorem 4.2.3 $M_t^u \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}_x)$ and its quadratic variation is given by

$$\langle M^u \rangle_t = \int_0^t \|\nabla u\|^2(X_s)\ ds.$$

We define quadratic covariation process by

$$\langle M^{u_1}, M^{u_2}\rangle = \frac{1}{2}\Big(\langle M^{u_1}+M^{u_2}\rangle - \langle M^{u_1}\rangle - \langle M^{u_2}\rangle\Big).$$

Lemma 4.2.6. *Let u_1, $u_2 \in C_0^\infty(E)$. Then*

$$\langle M^{u_1}, M^{u_2}\rangle_t = \int_0^t \langle \nabla u_1, \nabla u_2\rangle(X_s)\ ds.$$

Proof. By definition

$$\begin{aligned}
\langle M^{u_1}, M^{u_2}\rangle_t &= \frac{1}{2}\Big(\langle M^{u_1+u_2}\rangle_t - \langle M^{u_1}\rangle_t - \langle M^{u_2}\rangle_t\Big) \\
&= \frac{1}{2}\int_0^t \langle \nabla(u_1+u_2), \nabla(u_1+u_2)\rangle(X_s)\ ds - \frac{1}{2}\int_0^t \langle \nabla u_1, \nabla u_1\rangle(X_s)\ ds \\
&\quad -\frac{1}{2}\int_0^t \langle \nabla u_2, \nabla u_2\rangle(X_s)\ ds \\
&= \int_0^t \langle \nabla u_1, \nabla u_2\rangle(X_s)\ ds.
\end{aligned}$$

□

Theorem 4.2.7. *Under (HS1)-(HS3) after enlarging the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$ appropriately for every $x \in E$, the process $\mathbb{M}$ satisfies*

$$X_t = x + W_t + \int_0^t \left(\frac{\nabla\rho}{2\rho} + B\right)(X_s)\ ds, \quad t < \zeta \tag{4.14}$$

$\mathbb{P}_x$-a.s. for all $x \in E$ where W is a standard d-dimensional $(\mathcal{F}_t)$-Brownian motion on E. If additionally (HS4) holds, then we do not need to enlarge the stochastic basis and ζ can be replaced by ∞ (cf. Remark 4.1.13 and [42, Remark 3.4.3]).

Proof. Let $u_i \in C_0^\infty(E)$, $i = 1, \dots, d$. Note that by Lemma 4.2.6

$$\langle M^{u_i}, M^{u_j}\rangle_t = \int_0^t \langle \nabla u_i, \nabla u_j\rangle(X_s)\ ds, \quad 1 \leq i, j \leq d.$$

Suppose $\zeta < \infty$. Then it is standard that there is an enlargement $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}_x)$ (since $\langle \nabla u_i, \nabla u_j\rangle$ is degenerate) of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P}_x)$ and a d-dimensional Brownian motion $(W_t)_{t\geq 0} = (W_t^1, \dots, W_t^d)_{t\geq 0}$ on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}_x)$ and a $d \times d$ matrix $\sigma = (\sigma)_{1\leq i,j\leq d}$ such that

$$M_t^{u_i} = \sum_{k=1}^{d} \int_0^t \sigma_{ik}(X_s)\, dW_s^k, \quad 1 \leq i \leq d,\ t \geq 0.$$

and $\langle \nabla u_i, \nabla u_j \rangle = \sum_{k=1}^{d} \sigma_{ik}\sigma_{jk}$ (cf. [42, Theorem 4.2]). The identification of X up to ζ is now obtained by using Lemma 4.2.4 with an appropriate localizing sequence as in Lemma 4.1.11 for which the coordinate projections on E coincide locally with $C_0^\infty(E)$-functions and noting that $W_t^i = \int_0^t 1_E(X_s) dW_s^i$ on $\{t < \zeta\}$. If $\zeta = \infty$, then using the same localization, we obtain that $\langle M^{u_i} \rangle_t = \int_0^t 1_E(X_s)\, ds = t$ for $t < \infty$, where u_i is the i-th coordinate projection. Thus M^{u_i} is a Brownian motion by Lévy's characterization and we do not need an enlargement of stochastic basis. The localization of the drift part is trivial. □

4.3 Pathwise uniqueness and strong solutions

We first recall that by [43, Theorem 2.1] under the conditions (HS1), (HS2) ((HS3) is not needed), for every stochastic basis and given Brownian motion $(W_t)_{t\geq 0}$ there exists a strong solution to (4.14) which is pathwise unique among all solutions satisfying

$$\int_0^t \left\| \left(\frac{\nabla \rho}{2\rho} + B \right)(X_s) \right\|^2 \mathrm{d}s < \infty \quad \mathbb{P}_x\text{-a.s. on } \{t < \zeta\}\,. \tag{4.15}$$

In addition, one has pathwise uniqueness and weak uniqueness in this class.

In the situation of Theorem 4.2.7 it follows, however immediately from Lemma 4.2.4 that (4.15) holds for the solution there. Hence we obtain the following:

Theorem 4.3.1. *Assume (HS1)-(HS3). For every $x \in E$ the solution in Theorem 4.2.7 is strong, pathwise and weak unique. In particular, it is adapted to the filtration $(\mathcal{F}_t^W)_{t\geq 0}$ generated by the Brownian motion $(W_t)_{t\geq 0}$ in* (4.14).

Remark 4.3.2. *(i) By Theorem 4.2.7 and 4.3.1 we have thus shown that (the closure of)* (4.2) *is the Dirichlet form associated to the Markov processes given by the laws of the (strong) solutions to* (4.14)*. Hence we can use the theory of Dirichlet forms to show further properties of the solutions.*
(ii) In [43] also a new non-explosion criterion was proved (hence one obtains (HS4)),

assuming that $\frac{\nabla\rho}{2\rho}+B$ *is the (weak) gradient of a function* ψ *which is a kind of Lyapunov function for* (4.14)*. The theory of Dirichlet forms provides a number of analytic non-explosion, i.e. conservativeness criteria (hence implying (HS4)) which are completely different from the usual ones for SDEs and which are checkable in many cases. As stressed in (i) such criteria can now be applied to* (4.14)*. Even the simple already mentioned case, where* $m(\mathbb{R}^d)<\infty$ *and* $\|B\|\in L^1(\mathbb{R}^d,m)$ *which entails (HS4), appears to be a new non-explosion condition for* (4.14)*. Further explicit examples where* (4.14) *has a non-explosive unique strong solution are given in Section 4.4 below.*

4.4 Applications to Muckenhoupt A_β-weights

In this section we present a class of examples of ρ and B satisfying our assumptions (HS3) and (HS4). Throughout, we assume (HS1) and (HS2) to hold.

Lemma 4.4.1. *Suppose*

(i) For $r>0$

$$\left(\int_{B_r(0)}|u|^{\frac{2N}{N-2}}\ \rho\ dx\right)^{\frac{N-2}{2N}}\le c_r\left(\int_{B_{2r}(0)}\left(\|\nabla u\|^2+u^2\right)\ \rho\ dx\right)^{1/2},\quad \forall u\in C_0^\infty(\mathbb{R}^d),$$

where c_r *is some constant,* $N>2$ *and*

(ii) $\|B\|\in L^N_{loc}(\mathbb{R}^d,m)\cap L^\infty(K^c,m)$ *for some compact* $K\subset\mathbb{R}^d$.

Then

$$\left|\int_{\mathbb{R}^d}\langle B,\nabla u\rangle\ v\ \rho\ dx\right|\le c_{B,K}\ \mathcal{E}_1^0(u,u)^{1/2}\ \mathcal{E}_1^0(v,v)^{1/2},\quad \forall u,v\in C_0^\infty(\mathbb{R}^d),$$

where $c_{B,K}$ *is some constant, i.e. (HS3) holds.*

Proof. For $r_0 > 0$ such that $K \subset B_{r_0}(0)$

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^d} \langle B, \nabla u\rangle \; v \; \rho \; dx\right| \leq \left(\int_{\mathbb{R}^d} \|B\|^2 \; v^2 \; \rho \; dx\right)^{1/2} \; \left(\int_{\mathbb{R}^d} \|\nabla u\|^2 \rho \; dx\right)^{1/2} \\
\leq \;\; & \left(\int_{B_{r_0}(0)} \|B\|^2 \; v^2 \; \rho \; dx + \int_{B_{r_0}(0)^c} \|B\|^2 \; v^2 \; \rho \; dx\right)^{1/2} \; \mathcal{E}_1(u,u)^{1/2} \\
\leq \;\; & \left(\left(\int_{B_{r_0}(0)} \|B\|^2 \; v^2 \; \rho \; dx\right)^{1/2} + \|B\|_{\infty,K^c} \|v\|_{L^2(\mathbb{R}^d,m)}\right) \; \mathcal{E}_1(u,u)^{1/2} \\
\leq \;\; & \left(\left(\int_{B_{r_0}(0)} \|B\|^N \; \rho \; dx\right)^{1/N} \left(\int_{B_{r_0}(0)} v^{\frac{2N}{N-2}} \; \rho \; dx\right)^{\frac{N-2}{2N}} + \|B\|_{\infty,K^c} \|v\|_{L^2(\mathbb{R}^d,m)}\right) \\
& \cdot \mathcal{E}_1(u,u)^{1/2} \\
\leq \;\; & c_{B,K} \; \mathcal{E}_1^0(u,u)^{1/2} \mathcal{E}_1^0(v,v)^{1/2}
\end{aligned}
$$

The last inequality follows from assumption (i) and $\|\cdot\|_{\infty,K^c}$ denotes the $L^\infty(\mathbb{R}^d, m)$-norm on K^c. $\square$

Lemma 4.4.2. *Let ρ be a Muckenhoupt A_β-weight, $1 \leq \beta \leq 2$. Then for $x \in \mathbb{R}^d$, $r > 0$, $N > 2$*

$$
\left(\int_{B_r(x)} |u|^{\frac{2N}{N-2}} \; dm\right)^{\frac{N-2}{2N}} \leq C_{x,r} \left(\int_{B_{2r}(x)} \left(\|\nabla u\|^2 + u^2\right) \; dm\right)^{1/2}, \quad \forall u \in C^\infty(\mathbb{R}^d),
$$

where $C_{x,r}$ is some constant and $N \geq \beta d + \log_2 A$, A is the A_β constant of ρ.

Proof. By the doubling property of A_β-weights (cf. [72, Proposition 1.2.7]),

$$
m(B_{2r}(x)) \leq A \; 2^{\beta d} \; m(B_r(x)). \tag{4.16}
$$

Note that $A_\beta \subset A_2$ if $1 \leq \beta \leq 2$. Then by [26, Theorem (1.5)] the scaled Poincaré inequality holds true, i.e. for $x \in \mathbb{R}^d$, $r > 0$

$$
\int_{B_r(x)} |u - u_{x,r}|^2 \; dm \leq c r^2 \int_{B_r(x)} \|\nabla u\|^2 \; dm, \quad \forall u \in C^\infty(\mathbb{R}^d),
$$

where $u_{x,r} = \frac{1}{m(B_r(x))} \int_{B_r(x)} u \; dm$ and c is some constant. Consequently, [56, Theorem

2.1], the doubling property, and the scaled Poincaré inequality imply the Sobolev inequality, i.e. for $x \in \mathbb{R}^d$, $r > 0$, $N > 2$

$$\left(\int_{B_r(x)} |u|^{\frac{2N}{N-2}}\ dm\right)^{\frac{N-2}{2N}} \leq c_{x,r} \left(\int_{B_r(x)} \left(\|\nabla u\|^2 + u^2\right)\ dm\right)^{1/2}, \quad \forall u \in C_0^\infty(B_r(x)),$$

where $c_{x,r}$ is some constant and $N \geq \beta d + \log_2 A$. Then using a cutoff function like for instance $g_r(y) := \frac{1}{r}(2r - \|x-y\|)^+$, we see that for $x \in \mathbb{R}^d$, $r > 0$

$$\left(\int_{B_r(x)} |u|^{\frac{2N}{N-2}}\ dm\right)^{\frac{N-2}{2N}} \leq C_{x,r} \left(\int_{B_{2r}(x)} \left(\|\nabla u\|^2 + u^2\right)\ dm\right)^{1/2}, \quad \forall u \in C^\infty(\mathbb{R}^d),$$

where $C_{x,r}$ is some constant and $N > 2$ as well as $N \geq \beta d + \log_2 A$. □

Lemma 4.4.3. *Let ρ be a Muckenhoupt A_β weight, $1 \leq \beta \leq 2$, $N > 2$ and $\|B\| \in L^N_{loc}(\mathbb{R}^d, m) \cap L^\infty(K^c, m)$ for some compact $K \subset \mathbb{R}^d$, $N \geq \beta d + \log_2 A$, where A is the A_β constant of ρ. Then*

$$\left| \int_{\mathbb{R}^d} \langle B, \nabla u \rangle\ v\ \rho\ dx \right| \leq c_{B,K}\ \mathcal{E}^0_1(u,u)^{1/2}\ \mathcal{E}^0_1(v,v)^{1/2}, \quad \forall u, v \in C_0^\infty(\mathbb{R}^d),$$

where $c_{B,K}$ is some constant, i.e. (HS3) holds.

Proof. This follows from Lemma 4.4.1 and Lemma 4.4.2. □

Lemma 4.4.4. *It holds*

$$(1 - \hat{L})(C_0^\infty(\mathbb{R}^d)) \subset L^1(\mathbb{R}^d, m) \quad \textit{densely.}$$

In particular (HS4) holds (cf. Remark 4.1.8).

Proof. Let $h \in L^\infty(\mathbb{R}^d, m)$ be arbitrary. We have to show that

$$\int (1 - \hat{L}) f \cdot h\ dm = 0 \quad \forall f \in C_0^\infty(\mathbb{R}^d) \tag{4.17}$$

implies $h = 0$.
By [60, Theorem 2.1] it follows from (4.17) that $h \in D(\mathcal{E}^0)_{loc} := \{u \mid u \cdot \chi \in D(\mathcal{E}^0)\ \forall \chi \in$

$C_0^\infty(\mathbb{R}^d)\}$ and

$$\mathcal{E}_1^0(u,h) = -\int \langle B, \nabla u\rangle h \; dm \quad \forall u \in D(\mathcal{E}^0)_0 \tag{4.18}$$

where $D(\mathcal{E}^0)_0 := \{u \in D(\mathcal{E}^0) \mid \text{ supp}(|u|dm) \text{ is compact}\}$. Define

$$\begin{aligned}
v(r) : &= m(B_r(0)), \quad r > 0 \\
a_n : &= \int_n^{2n} \frac{s}{\log(v(s))} \; ds, \quad n \geq 1 \\
\psi_n(r) : &= 1_{[0,n]}(r) - \frac{1}{a_n}\int_n^r \frac{s}{\log(v(s))} \; ds \cdot 1_{[n,2n]}(r) \\
u_n(x) : &= \psi_n(\|x\|).
\end{aligned}$$

Then $u_n \in D(\mathcal{E}^0)_0$ and

$$\nabla u_n(x) = -\frac{1}{a_n}\frac{x}{\log(v(\|x\|))} \cdot 1_{[n,2n]}(\|x\|) \tag{4.19}$$

$$a_n \geq \int_n^{2n} \frac{n}{\log(v(2n))} \; ds = \frac{n^2}{\log(v(2n))} \geq \frac{n^2}{\log(A2^{\beta d} \; v(n))}. \tag{4.20}$$

The last inequality follows from (4.16). Taking sufficiently large n such that $\log(A2^{\beta d}) \leq \log(v(n))$, (4.19) and (4.20) imply

$$\|\nabla u_n(x)\| \leq \frac{\log(A2^{\beta d} \; v(n))}{n^2} \; \frac{2n}{\log(v(n))} \cdot 1_{[n,2n]}(\|x\|) \leq \frac{4}{n} \cdot 1_{[n,2n]}(\|x\|). \tag{4.21}$$

Then

$$\begin{aligned}
\phi(n) : &= \int_{B_n(0)} h^2 \; dm \leq \int_{B_{2n}(0)} h^2 u_n^2 \; dm = \int_{B_{2n}(0)} (hu_n^2) \cdot h \; dm \\
&= -\int_{B_{2n}(0)} \langle \nabla(hu_n^2), \nabla h\rangle \; dm - \int_{B_{2n}(0)} \langle B, \nabla(hu_n^2)\rangle h \; dm
\end{aligned}$$

Since $hu_n^2 \in D(\mathcal{E}^0)_0$, the last equality follows from (4.18). The last term is equal to

$$\begin{aligned}
& -\int_{B_{2n}(0)} \langle B, \nabla(u_n \cdot (hu_n))\rangle h \; dm \\
= & -\int_{B_{2n}(0)} \langle B, \nabla(u_n)\rangle h^2 u_n \; dm - \int_{B_{2n}(0)} \langle B, \nabla(hu_n)\rangle hu_n \; dm.
\end{aligned}$$

Since $hu_n \in D(\mathcal{E}^0)_0$, the second term is zero by (HS3). Therefore

$$\begin{aligned}
\phi(n) &\leq -\int_{B_{2n}(0)} \langle \nabla(hu_n^2), \nabla h\rangle \; dm - \int_{B_{2n}(0)} \langle B, \nabla(u_n)\rangle h^2 u_n \; dm \\
&= -\int_{B_{2n}(0)} \left(u_n^2 \|\nabla h\|^2 + 2u_n \langle \nabla h, \nabla u_n\rangle h\right) \; dm - \int_{B_{2n}(0)} \langle B, \nabla(u_n)\rangle h^2 u_n \; dm \\
&\leq \int_{B_{2n}(0)} \|\nabla u_n\|^2 h^2 \; dm - \int_{B_{2n}(0)} \langle B, \nabla(u_n)\rangle h^2 u_n \; dm.
\end{aligned}$$

Taking $n \geq 4$ so large that $K \subset B_n(0)$ and that (4.21) holds

$$\begin{aligned}
\phi(n) &\leq \left(\frac{4}{n}\right)^2 \int_{B_{2n}(0)\setminus B_n(0)} h^2 \; dm + \frac{4}{n}\|B\|_{\infty, K^c} \int_{B_{2n}(0)\setminus B_n(0)} h^2 \; dm \\
&\leq \frac{4}{n}(\|B\|_{\infty,K^c} + 1) \int_{B_{2n}(0)} h^2 \; dm = \frac{4}{n}(\|B\|_{\infty,K^c} + 1)\phi(2n).
\end{aligned}$$

Set $C := 4(\|B\|_{\infty,K^c} + 1)$. Thus by iteration of the last inequality and (4.16), we obtain for any $k \geq 1$

$$\phi(n) \leq \frac{C^k}{n^k 2^{\frac{k(k+1)}{2}}} \; \phi(2^k n) \leq \frac{C^k}{n^k 2^{\frac{k(k+1)}{2}}} \; \|h\|_\infty^2 \; v(2^k n) \leq \frac{C^k}{n^k 2^{\frac{k(k+1)}{2}}} \|h\|_\infty^2 \; (A2^{\beta d})^k \; v(n).$$

Note that $v(n) \leq cn^\alpha$ for some $\alpha > 0$, where $c > 0$ is some constant. Now choose $k > \alpha$ then $\phi(n) \to 0$ as $n \to \infty$, hence $h = 0$. □

Lemma 4.4.3 and Lemma 4.4.4 imply the final theorem.

Theorem 4.4.5. *Let ρ and B satisfy the assumptions (HS1) and (HS2) and the assumptions of Lemma 4.4.3. Then (HS1)-(HS4) hold. Consequently, Theorems 4.2.7 and 4.3.1 apply with $\zeta = \infty$.*

Chapter 5

Degenerate elliptic forms w.r.t. 2-admissible weights

5.1 Preliminaries and degenerate elliptic forms with respect to 2-admissible weights

We say ρ is 2-admissible if the following four conditions are satisfied (see [41, Section 1.1]):

(I) $0 < \rho(x) < \infty$ dx-a.e. $x \in \mathbb{R}^d$ and ρ is doubling, i.e. there is a constant $C_1 > 0$ such that for any ball $B_r(x) \subset \mathbb{R}^d$

$$m(B_{2r}(x)) \le C_1\, m(B_r(x)), \quad m := \rho dx. \tag{5.1}$$

(II) If $D \subset \mathbb{R}^d$ is an open set and $(u_n)_{n\ge 1} \subset C^\infty(D)$ is a sequence of functions such that

$$\lim_{n\to\infty} \int_D |u_n|^2\, dm = 0, \quad \text{and} \quad \lim_{n\to\infty} \int_D \|\nabla u_n - \vartheta\|^2\, dm = 0,$$

then $\vartheta = 0$.

(III) There are constants $\theta > 1$ and $C_2 > 0$ such that for $x \in \mathbb{R}^d$ and $r > 0$

$$\left(\frac{1}{m(B_r(x))} \int_{B_r(x)} |u|^{2\theta}\, dm \right)^{1/2\theta} \le C_2\, r \left(\frac{1}{m(B_r(x))} \int_{B_r(x)} \|\nabla u\|^2\, dm \right)^{1/2},$$

for all $u \in C_0^\infty(B_r(x))$.

(IV) There is a constant $C_3 > 0$ such that for $x \in \mathbb{R}^d$ and $r > 0$

$$\int_{B_r(x)} |u - u_{x,r}|^2 \, dm \le C_3 \, r^2 \int_{B_r(x)} \|\nabla u\|^2 \, dm, \quad \forall u \in C^\infty(B_r(x)) \cap C_b(B_r(x)),$$

where $u_{x,r} = \frac{1}{m(B_r(x))} \int_{B_r(x)} u \, dm$.

Remark 5.1.1. *We know from [39, Theorem 13.1] that a locally integrable weight ρ is a 2-admissible weight if and only if ρ is doubling and there exist constants $c > 0$, $\gamma \ge 1$ such that for $x \in \mathbb{R}^d$ and $r > 0$*

$$\frac{1}{m(B_r(x))} \int_{B_r(x)} |u - u_{x,r}| \, dm \le c \, r \left(\frac{1}{m(B_{\gamma r}(x))} \int_{B_{\gamma r}(x)} \|\nabla u\|^2 \, dm \right)^{1/2},$$

whenever $u \in C^\infty(B_{\gamma r}(x))$ (weak (1,2) Poincaré inequality).

Throughout this chapter let ρ be a locally integrable 2-admissible weight. For later use we define a symmetric bilinear form

$$\mathcal{E}^\rho(f,g) = \frac{1}{2} \int_{\mathbb{R}^d} \langle \nabla f, \nabla g \rangle \, dm, \quad f, g \in C_0^\infty(\mathbb{R}^d). \tag{5.2}$$

By (II) $(\mathcal{E}^\rho, C_0^\infty(\mathbb{R}^d))$ is closable in $L^2(\mathbb{R}^d, m)$ and its closure $(\mathcal{E}^\rho, D(\mathcal{E}^\rho))$ is a strongly local, regular Dirichlet form in the sense of [35].

Consider the following assumption:

(HP1) Let $A = (a_{ij})_{1 \le i,j \le d}$ be a symmetric possibly degenerate (uniformly weighted) elliptic $d \times d$ matrix, that is $a_{ij} \in L^1_{loc}(\mathbb{R}^d, dx)$ and there exists a constant $\lambda \ge 1$ such that for dx-a.e. $x \in \mathbb{R}^d$

$$\lambda^{-1} \, \rho(x) \, \|\xi\|^2 \le \langle A(x)\xi, \xi \rangle \le \lambda \, \rho(x) \, \|\xi\|^2, \quad \forall \xi \in \mathbb{R}^d. \tag{5.3}$$

From now on, we fix A satisfying (HP1) and consider the symmetric bilinear form

$$\mathcal{E}^A(f,g) = \frac{1}{2} \int_{\mathbb{R}^d} \langle A \, \nabla f, \nabla g \rangle \, dx, \quad f, g \in C_0^\infty(\mathbb{R}^d).$$

By closability of $(\mathcal{E}^\rho, C_0^\infty(\mathbb{R}^d))$ in $L^2(\mathbb{R}^d, m)$ and (5.3) $(\mathcal{E}^A, C_0^\infty(\mathbb{R}^d))$ is closable in $L^2(\mathbb{R}^d, m)$. The closure $(\mathcal{E}^A, D(\mathcal{E}^A))$ of $(\mathcal{E}^A, C_0^\infty(\mathbb{R}^d))$ is a strongly local, regular, symmetric Dirichlet form (see [44, II. Section 2 b) and c)]). The Dirichlet form $(\mathcal{E}^A, D(\mathcal{E}^A))$

can be written as

$$\mathcal{E}^A(f,g) = \frac{1}{2}\int_{\mathbb{R}^d} d\Gamma^A(f,g), \quad f,g \in D(\mathcal{E}^A),$$

where Γ^A is a symmetric bilinear form on $D(\mathcal{E}^A) \times D(\mathcal{E}^A)$ with values in the signed Radon measures on $\mathbb{R}^d$ (called energy measures). We can extend the quadratic form $u \mapsto \Gamma^A(u,u)$ to $D(\mathcal{E}^A)_{loc} = \{u \in L^2_{loc}(\mathbb{R}^d, m) \mid \Gamma^A(u,u) \text{ is a Radon measure}\}$, where $D(\mathcal{E}^A)_{loc}$ is the set of all m-measurable functions u on $\mathbb{R}^d$ for which on every compact set $K \subset \mathbb{R}^d$ there exists a function $v \in D(\mathcal{E}^A)$ with $u = v$ m-a.e on K (cf. [63, p. 274]). Then the energy measure Γ^A defines in an intrinsic way a pseudo metric d on $\mathbb{R}^d$ by

$$d(x,y) = \sup\Big\{u(x) - u(y) \mid u \in D(\mathcal{E}^A)_{loc} \cap C(\mathbb{R}^d),\ \Gamma^A(u,u) \le m \text{ on } \mathbb{R}^d\Big\},$$

where $\Gamma^A(u,u) \le m$ means that the energy measure $\Gamma^A(u,u)$ is absolutely continuous w.r.t. the reference measure m with Radon-Nikodym derivative $\frac{d}{dm}\Gamma^A(u,u) \le 1$. We define the balls w.r.t. intrinsic metric by

$$\tilde{B}_r(x) = \{y \in \mathbb{R}^d \mid d(x,y) < r\}, \quad x \in \mathbb{R}^d, \quad r > 0.$$

Let $(T_t)_{t>0}$ and $(G_\alpha)_{\alpha>0}$ be the $L^2(\mathbb{R}^d, m)$-semigroup and resolvent associated to $(\mathcal{E}^A, D(\mathcal{E}^A))$ and $(L, D(L))$ be the corresponding generator (see [44, Diagram 3, p. 39]).

We assume from now on

(HP2) Either $\rho \in H^{1,1}_{loc}(\mathbb{R}^d, dx)$ or $\rho^{-1} \in L^1_{loc}(\mathbb{R}^d, dx)$.

Lemma 5.1.2. *For any $x, y \in \mathbb{R}^d$*

$$\frac{1}{\sqrt{\lambda}}\ \|x-y\| \le d(x,y) \le \sqrt{\lambda}\ \|x-y\|, \tag{5.4}$$

where $\lambda \in [1,\infty)$ is as in (5.3).

Proof. For any $z \in \mathbb{R}^d$ the map

$$u \ :\ x \longmapsto u(x) := \langle x, z\rangle$$

lies in $D(\mathcal{E}^A)_{loc} \cap C(\mathbb{R}^d)$. For fixed $y, y' \in \mathbb{R}^d$ $(y \neq y')$ and $\lambda \in [1, \infty)$, choose

$$z = \frac{(y - y')}{\sqrt{\lambda}\, \|y - \ y'\|} \in \mathbb{R}^d.$$

Then by (5.3)

$$\int_B \Gamma^A(u,u) = \int_B \langle A\nabla u, \nabla u\rangle dx \leq \lambda \int_B \|\nabla u\|^2 \rho\ dx = \int_B \rho dx, \quad \forall B \in \mathcal{B}(\mathbb{R}^d).$$

Hence $\Gamma^A(u,u) \leq m$. Furthermore

$$u(y) - u(y') = \frac{1}{\sqrt{\lambda}}\ \|y - y'\|.$$

Therefore for any $x, y \in \mathbb{R}^d$

$$d(x,y) \geq \frac{1}{\sqrt{\lambda}}\ \|x - y\|.$$

Define

$$d^\rho(x,y) = \sup\Big\{u(x) - u(y) \mid u \in D(\mathcal{E}^\rho)_{loc} \cap C(\mathbb{R}^d),\ \Gamma^\rho(u,u) \leq m \text{ on } \mathbb{R}^d\Big\}.$$

Here (cf. (5.2)) Γ^ρ is a symmetric bilinear form on $D(\mathcal{E}^\rho) \times D(\mathcal{E}^\rho)$ such that

$$\mathcal{E}^\rho(f,g) = \frac{1}{2}\int_{\mathbb{R}^d}\ d\Gamma^\rho(f,g).$$

By [65, Theorem 4.1] and (HP2)

$$d^\rho(x,y) = \|x - y\|, \quad \forall x, y \in \mathbb{R}^d.$$

Note that by (5.3) $D(\mathcal{E}^A) = D(\mathcal{E}^\rho)$. Fix $x,\ y \in \mathbb{R}^d$. Suppose $(u_n)_{n\geq 1} \subset D(\mathcal{E}^A)_{loc} \cap C(\mathbb{R}^d)$ with $\Gamma^A(u_n,u_n) \leq m$ on $\mathbb{R}^d$ be a sequence such that

$$d(x,y) = \lim_{n\to\infty}\Big(u_n(x) - u_n(y)\Big).$$

Since $\Gamma^\rho\big(u_n/\sqrt{\lambda}, u_n/\sqrt{\lambda}\big) = \lambda^{-1}\Gamma^\rho(u_n,u_n) \leq \Gamma^A(u_n,u_n) \leq m$, by definition of $d^\rho(x,y)$

$$\|x - y\| = d^\rho(x,y) \geq \frac{1}{\sqrt{\lambda}}\lim_{n\to\infty}\Big(u_n(x) - u_n(y)\Big) = \frac{1}{\sqrt{\lambda}} d(x,y).$$

□

Remark 5.1.3. *(i) Assumption (HP1) is only used in order to show the (strong) equivalence of the intrinsic metric $d(\cdot,\cdot)$ and the Euclidean metric as in (5.4).*

(ii) Note that $\tilde{B}_r(x) \in \mathcal{B}(\mathbb{R}^d)$ for any $x \in \mathbb{R}^d$, $r > 0$, since the (strong) equivalence (5.4) between two metrics implies topological equivalence.

The doubling property w.r.t. the intrinsic metric $d(\cdot,\cdot)$ holds true, if λ takes values in $[1,2)$:

Lemma 5.1.4. *Let $\lambda \in [1,2)$. There exists a constant $c > 0$ such that for any $x \in \mathbb{R}^d$, $r > 0$*

$$m(\tilde{B}_{2r}(x)) \leq c\ m(\tilde{B}_r(x)). \tag{5.5}$$

Proof. Let $x \in \mathbb{R}^d$, $r > 0$. By (5.1) and (5.4)

$$m(\tilde{B}_{2r}(x)) \leq m(B_{2\sqrt{\lambda}r}(x)) \leq C_1\ m(B_{\sqrt{\lambda}r}(x)) \leq C_1 m(\tilde{B}_{\lambda r}(x)),$$

where C_1 is the constant as in (5.1). Iterating the above inequality, we get

$$m(\tilde{B}_{2r}(x)) \leq C_1^n\ m(\tilde{B}_{2r(\frac{\lambda}{2})^n}(x)), \quad \forall n \in \mathbb{N}.$$

Since $\left(\frac{\lambda}{2}\right)^n \to 0$ as $n \to \infty$, we can find a constant $c > 0$ independent of x, r such that

$$m(\tilde{B}_{2r}(x)) \leq c\ m(\tilde{B}_r(x)).$$

□

In view of Lemma 5.1.4 from now on, we assume

(HP3) $\lambda \in [1,2)$.

Lemma 5.1.5. *Let A_1, $A_2 \in \mathcal{B}(\mathbb{R}^d)$ be bounded sets with $A_1 \subset A_2$. Then*

$$\int_{A_1} |u - u_{A_1}|^2\ dm \leq \int_{A_2} |u - u_{A_2}|^2\ dm, \quad u \in C(\mathbb{R}^d),$$

where $u_B = \frac{1}{m(B)} \int_B u\ dm$, $B \in \mathcal{B}(\mathbb{R}^d)$.

Proof. Let $u \in C(\mathbb{R}^d)$. Then

$$\begin{aligned} & \int_{A_1} |u - u_{A_2}|^2 \, dm - \int_{A_1} |u - u_{A_1}|^2 \, dm \\ = \; & -2u_{A_2} \int_{A_1} u \, dm + u_{A_2}^2 \, m(A_1) + 2u_{A_1} \int_{A_1} u \, dm - u_{A_1}^2 \, m(A_1) \\ = \; & -2u_{A_2} u_{A_1} \, m(A_1) + u_{A_2}^2 \, m(A_1) + 2u_{A_1}^2 \, m(A_1) - u_{A_1}^2 \, m(A_1) \\ = \; & (u_{A_2} - u_{A_1})^2 \, m(A_1) \geq 0. \end{aligned}$$

Since $A_1 \subset A_2$,

$$\int_{A_1} |u - u_{A_1}|^2 \, dm \leq \int_{A_2} |u - u_{A_2}|^2 \, dm.$$

□

The scaled weak Poincaré inequality w.r.t. the intrinsic metric $d(\cdot, \cdot)$ holds true:

Lemma 5.1.6. *For $x \in \mathbb{R}^d$, $r > 0$*

$$\int_{\tilde{B}_r(x)} |u - \tilde{u}_{x,r}|^2 \, dm \leq c \, r^2 \int_{\tilde{B}_{2r}(x)} d\Gamma^A(u,u), \quad u \in D(\mathcal{E}^A),$$

where $c > 0$ is a constant and $\tilde{u}_{x,r} = \frac{1}{m(\tilde{B}_r(x))} \int_{\tilde{B}_r(x)} u \, dm$.

Proof. By (IV) and (5.3) for $x \in \mathbb{R}^d$, $r > 0$

$$\begin{aligned} \int_{B_r(x)} |u - u_{x,r}|^2 \, dm & \leq C_3 r^2 \int_{B_r(x)} \|\nabla u\|^2 \, dm, \\ & \leq \lambda \, C_3 r^2 \int_{B_r(x)} d\Gamma^A(u,u), \quad \forall u \in C^\infty(\mathbb{R}^d), \end{aligned} \tag{5.6}$$

where C_3 is the constant as in (IV) and λ is the constant of (HP1) satisfying (HP3). Therefore by Lemma 5.1.5, (5.4), and (5.6)

$$\begin{aligned} \int_{\tilde{B}_r(x)} |u - \tilde{u}_{x,r}|^2 \, dm & \leq \int_{B_{\sqrt{\lambda} r}(x)} |u - u_{x,\sqrt{\lambda} r}|^2 \, dm \leq \lambda^2 \, C_3 r^2 \int_{B_{\sqrt{\lambda} r}(x)} d\Gamma^A(u,u) \\ & \leq \lambda^2 \, C_3 r^2 \int_{\tilde{B}_{2r}(x)} d\Gamma^A(u,u), \quad \forall u \in C^\infty(\mathbb{R}^d). \end{aligned}$$

□

Remark 5.1.7. *By [63, Theorem 2.4] and Lemma 5.1.2, 5.1.4, 5.1.6, the scaled strong Poincaré inequality holds true, i.e. for $x \in \mathbb{R}^d$, $r > 0$*

$$\int_{\tilde{B}_r(x)} |u - \tilde{u}_{x,r}|^2 \, dm \leq c^\star \, r^2 \int_{\tilde{B}_r(x)} d\Gamma^A(u,u), \quad u \in D(\mathcal{E}^A),$$

where $c^\star > 0$ is some constant.

Theorem 5.1.8. *The Dirichlet form $(\mathcal{E}^A, D(\mathcal{E}^A))$ is conservative.*

Proof. By the doubling property (5.5) and [38, Proposition 5.1, 5.2]

$$c_1 \, r^{\alpha'} \leq m(\tilde{B}_r(0)) \leq c_2 \, r^{\alpha}, \quad \forall r \geq 1,$$

where $c_1, c_2, \alpha, \alpha' > 0$ are some constants. Therefore,

$$\int_1^\infty \frac{r}{\log\big(m(\tilde{B}_r(0))\big)} \, dr = \infty.$$

Then by [65, Theorem 3.6] and Lemma 5.1.2 the conservativeness follows (cf. Proposition 2.1.4). □

By Lemma 5.1.2, 5.1.4, 5.1.6 the properties (Ia)-(Ic) of [63] are satisfied. Therefore by [63, p. 286 A)] there exists a jointly continuous transition kernel density $p_t(x,y)$ such that

$$P_t f(x) := \int_{\mathbb{R}^d} p_t(x,y) \, f(y) \; m(dy), \quad t > 0, \; x, y \in \mathbb{R}^d, \; f \in \mathcal{B}_b(\mathbb{R}^d)$$

is an m-version of $T_t f$ if $f \in L^2(\mathbb{R}^d, m)_b$. Throughout this chapter we set $P_0 := id$. Taking the Laplace transform of $p_\cdot(x,y)$, we obtain a $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d)$ measurable non-negative resolvent kernel density $r_\alpha(x,y)$ such that

$$R_\alpha f(x) := \int_{\mathbb{R}^d} r_\alpha(x,y) \, f(y) \, m(dy) \, , \; \alpha > 0, \; x \in \mathbb{R}^d, f \in \mathcal{B}_b(\mathbb{R}^d),$$

is an m-version of $G_\alpha f$ if $f \in L^2(\mathbb{R}^d, m)_b$. For a signed Radon measure μ on $\mathbb{R}^d$, let us define

$$R_\alpha \mu(x) = \int_{\mathbb{R}^d} r_\alpha(x,y) \, \mu(dy) \, , \quad \alpha > 0, \quad x \in \mathbb{R}^d,$$

whenever this makes sense.

Theorem 5.1.9. *For $x, y \in \mathbb{R}^d$, $t > 0$ and any $\varepsilon > 0$*

$$p_t(x,y) \le c\ m(B_{\sqrt{t}}(x))^{-1/2}\ m(B_{\sqrt{t}}(y))^{-1/2}\ \exp\left(-\frac{\|x-y\|^2}{\lambda(4+\varepsilon)t}\right), \tag{5.7}$$

where c is some constant and λ is the constant of (HP1) satisfying (HP3).

Proof. It follows from [63, Corollary 4.2 and Remarks (ii) in p.286] that for $x, y \in \mathbb{R}^d$, $t > 0$ and any $\varepsilon > 0$

$$p_t(x,y) \le c_1\ m\big(\tilde{B}_{\sqrt{t}}(x)\big)^{-1/2} m\big(\tilde{B}_{\sqrt{t}}(y)\big)^{-1/2}\ \exp\left(-\frac{d(x,y)^2}{(4+\varepsilon)t}\right),$$

where c_1 is some constant. By (5.1) and Lemma 5.1.2 the assertion then follows. □

Proposition 5.1.10. *It holds:*

(i) $(P_t)_{t\ge 0}$ *(resp. $(R_\alpha)_{\alpha>0}$) is strong Feller, i.e. for $t > 0$, we have $P_t(\mathcal{B}_b(\mathbb{R}^d)) \subset C_b(\mathbb{R}^d)$ (resp. for $\alpha > 0$, we have $R_\alpha(\mathcal{B}_b(\mathbb{R}^d)) \subset C_b(\mathbb{R}^d)$).*

(ii) $P_t(L^1(\mathbb{R}^d, m)_0) \subset C_\infty(\mathbb{R}^d)$.

Proof. Using the transition density estimate (5.7), the statements follow exactly as in Proposition 3.2.3. □

In order to introduce conveniently some notations, we suppose up to the end of this section that there exists a Hunt process

$$\mathbb{M} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \zeta, (X_t)_{t\ge 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d\cup\{\Delta\}}) \tag{5.8}$$

associated with the transition function $(P_t)_{t\ge 0}$.

Remark 5.1.11. *Under the existence of Hunt process* (5.8)*, by Theorem 5.1.8 and Proposition 5.1.10 (i), $\mathbb{P}_x(\zeta = \infty) = 1$ for all $x \in \mathbb{R}^d$.*

Let D be an open set in $\mathbb{R}^d$. Then the part Dirichlet form $(\mathcal{E}^{A,D}, D(\mathcal{E}^{A,D}))$ of $(\mathcal{E}^A, D(\mathcal{E}^A))$ on D is a regular Dirichlet form on $L^2(D, m)$ (cf. [35, Section 4.4]). We refer to subsection 3.1.2 for more definitions about part Dirichlet form and 1-potential. We only note that for $\omega \in \Omega$

$$X_t^D(\omega) := \begin{cases} X_t(\omega) & 0 \le t < D_{D^c}(\omega) \\ \Delta & t \ge D_{D^c}(\omega). \end{cases} \tag{5.9}$$

Proposition 5.1.12. *Let μ be a positive Radon measure and $G \subset \mathbb{R}^d$ relatively compact open. Suppose*

$$\int_G r_1(\cdot, y)\ \mu(dy) \le r_1^G$$

μ-a.e. on G and m-a.e. on $\mathbb{R}^d$, where r_1^G is a continuous function on $\mathbb{R}^d$. Then $1_G \cdot \mu \in S_{00}$.

Proof. Since $R_1(1_G \cdot \mu)(x) = \int_G r_1(x, y)\ \mu(dy) \le r_1^G(x)$ for μ-a.e. $x \in G$, $R_1(1_G \cdot \mu) \in L^1(G, \mu)$. This implies that $1_G \cdot \mu \in S_0$ by [35, Example 4.2.2]. Then $1_G \cdot \mu \in S_{00}$ by Proposition 3.1.14. □

5.2 Concrete 2-admissible weights with polynomial growth

Definition 5.2.1. *(i) A function $\psi \in \mathcal{B}(\mathbb{R}^d)$ with $\psi > 0$ dx-a.e. is said to be a Muckenhoupt A_2-weight (in notation $\psi \in A_2$), if there exists a positive constant A such that, for every ball $B \subset \mathbb{R}^d$,*

$$\left(\int_B \psi dx\right)\left(\int_B \psi^{-1} dx\right) \le A\left(\int_B 1\, dx\right)^2.$$

(ii) A mapping $F : \mathbb{R}^d \to \mathbb{R}^d$ is said to be quasi-conformal if F is one-to-one, the components F_i, $i = 1, \cdots, d$ of F have distributional derivatives belonging to $L^d_{loc}(\mathbb{R}^d, dx)$ and there is a constant $M > 0$ called dilation constant of F, so that dx-a.e.

$$\left(\sum_{1 \le i,j \le d} \left(\partial_j F_i\right)^2\right)^{1/2} \le M\ \left|\det\ F'\right|^{1/d},$$

where $F'(x) = (\partial_j F_i(x))_{1 \le i,j \le d}$.

2-admissible weights arise typically as in the following example:

Example 5.2.2. *(cf. [41, Chapter 15])*

(1) If $\rho \in A_2$, then ρ is a 2-admissible weight.

(2) If $\rho(x) = |\det F'(x)|^{1-2/d}$ where F is a quasi-conformal mapping in $\mathbb{R}^d$, then ρ is a 2-admissible weight.

In this section we consider

$$\rho(x) = \|x\|^{\alpha}, \quad \alpha \in (-d, \infty), \quad d \geq 2. \tag{5.10}$$

Note that for any $\alpha \in (-d, \infty)$ $\rho(x) = \|x\|^{\alpha}$ satisfies (HP2). In particular, if $\alpha \in (-d, d)$, then $\rho \in A_2$ (see [72, Example 1.2.5]) and if $\alpha \in (-d+2, \infty)$, then $\rho = |\mathrm{det} F'|^{1-2/d}$ for some quasi-conformal mapping F (cf. [26, Section 3]). Thus ρ as in (5.10) is a 2-admissible weight by Example 5.2.2.

Lemma 5.2.3. *Let ρ be as in* (5.10). *Then:*

(i) $\lim_{t \to 0} P_t f(x) = f(x)$ *for each* $x \in \mathbb{R}^d$ *and* $f \in C_0(\mathbb{R}^d)$.

(ii) $P_t C_0(\mathbb{R}^d) \subset C_\infty(\mathbb{R}^d)$ *for each* $t > 0$.

In particular, $(P_t)_{t \geq 0}$ *is a Feller semigroup.*

Proof. By Proposition 5.1.10 (ii), $P_t C_0(\mathbb{R}^d) \subset C_\infty(\mathbb{R}^d)$ for each $t > 0$. Note that for $\alpha \in [0, \infty)$ and $0 < \sqrt{t} \leq \|x\|$, we have

$$m(B_{\sqrt{t}}(x)) \geq c_d \; (\|x\| - \sqrt{t})^{\alpha} \; \sqrt{t}^{d},$$

with $c_d = vol(B_1(0))$. Then the statement (i) can be derived as in the proof of Lemma 3.2.6 (i), using Theorem 5.1.9, the conservativeness of $(\mathcal{E}^A, D(\mathcal{E}^A))$ and the transition density estimate (5.7). Hence by Lemma 3.1.3 $(P_t)_{t \geq 0}$ is a Feller semigroup. $\square$

In view of Lemma 5.2.3 and the classical Feller theory, there exists a Hunt process

$$\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \zeta, (X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}^d_\Delta}),$$

with state space $\mathbb{R}^d$ and lifetime ζ such that $P_t(x, B) := P_t 1_B(x) = \mathbb{P}_x(X_t \in B)$ for any $x \in \mathbb{R}^d$, $B \in \mathcal{B}(\mathbb{R}^d)$, $t \geq 0$. Thus the existence of $\mathbb{M}$ as in (5.8) is guaranteed. As usual any function $f : \mathbb{R}^d \to \mathbb{R}$ is extended to $\{\Delta\}$ by setting $f(\Delta) := 0$.

5.2.1 Concrete Muckenhoupt A_2-weights with polynomial growth

In this subsection, we consider the case where ρ as in (5.10) belongs to A_2. More precisely, we consider

$$\rho(x) = \|x\|^{\alpha}, \quad \alpha \in (-d, 2), \quad d \geq 3. \tag{5.11}$$

Lemma 5.2.4. *Let ρ be as in (5.11). Then*

$$r_1(x,y) \le c_1 \left(\Phi(x,y) + \Psi(x,y) 1_{\{\alpha\in(-d,0)\}}\right), \tag{5.12}$$

where $\Phi(x,y) := \frac{1}{\|x-y\|^{\alpha+d-2}}$, $\Psi(x,y) := \frac{1}{\|x-y\|^{d-2}\|y\|^{\alpha}}$, and c_1 is some constant.

Proof. Note that for $\alpha \in [0,2)$, $t > 0$, and $x \in \mathbb{R}^d$, we have

$$c_2\ \sqrt{t}^{\alpha+d} \le m(B_{\sqrt{t}}(x)) \le c_3\ \sqrt{t}^{d}\ (\|x\| + \sqrt{t})^{\alpha},$$

where c_2, c_3 are some constants. Then the assertion follows as in the proof of Lemma 3.2.6 (ii) using the transition density estimate (5.7). □

Define

$$V_\eta g(x) := \int_{\mathbb{R}^d} \frac{1}{\|x-y\|^{d-\eta}}\ g(y)\ dy, \quad x \in \mathbb{R}^d,\ \eta > 0,$$

whenever it makes sense.

Lemma 5.2.5. *Let $\eta \in (0,d)$, $0 < \eta - \frac{d}{p} < 1$ and $g \in L^p(\mathbb{R}^d, dx)$ with*

$$\int_{\mathbb{R}^d} (1 + \|y\|)^{\eta-d} |g(y)|\ dy < \infty.$$

Then $V_\eta g$ is Hölder continuous of order $\eta - \frac{d}{p}$.

Proof. See [45, Chapter 4, Theorem 2.2]. □

Lemma 5.2.6. *Let ρ be as in (5.11) and $G \subset \mathbb{R}^d$ any relatively compact open set, $p \ge 1$. Suppose*

(i) if $\alpha \in (-d, -d+2]$, $1_G \cdot f\ \|x\|^{\alpha} \in L^1(\mathbb{R}^d, dx)$ and $1_G \cdot f \in L^q(\mathbb{R}^d, dx)$, $0 < 2 - \frac{d}{q} < 1$,

(ii) if $\alpha \in (-d+2, 0)$, $1_G \cdot f\ \|x\|^{\alpha} \in L^p(\mathbb{R}^d, dx)$ with $0 < 2 - \alpha - \frac{d}{p} < 1$ and $1_G \cdot f \in L^q(\mathbb{R}^d, dx)$ with $0 < 2 - \frac{d}{q} < 1$,

(iii) if $\alpha \in [0,2)$, $1_G \cdot f\ \|x\|^{\alpha} \in L^p(\mathbb{R}^d, dx)$ with $0 < 2 - \alpha - \frac{d}{p} < 1$.

Then $R_1(1_G \cdot |f|m)$ is bounded everywhere (hence clearly also bounded m-a.e. on $\mathbb{R}^d$ and $R_1(1_G \cdot |f|m) \in L^1(G, |f|m)$) by the continuous function $\int_G |f(y)|\ (\Phi(\cdot,y) + \Psi(\cdot,y)\ 1_{\{\alpha\in(-d,0)\}})\ m(dy)$. In particular, $1_G \cdot |f|m \in S_{00}$.

Proof. By Lemma 5.2.4 for any $x \in \mathbb{R}^d$

$$\begin{aligned} R_1(1_G \cdot |f|m)(x) &\le c_1 \int_G |f(y)| \, \left(\Phi(x,y) + \Psi(x,y) 1_{\{\alpha\in(-d,0)\}}\right) \, m(dy) \\ &= c_1\Big(\, V_{2-\alpha}(1_G \cdot |f| \, \|y\|^{\alpha})(x) + V_2(1_G \cdot |f|)(x) 1_{\{\alpha\in(-d,0)\}}\Big). \end{aligned}$$

where c_1 is the constant as in (5.12). If $\alpha \in (-d, -d+2]$, clearly $\int_G |f(y)| \, \Phi(\cdot, y) \, m(dy)$ is continuous. Furthermore, $\int_G |f(y)| \psi(\cdot, y) \, m(dy)$ is continuous by Lemma 5.2.5. By Proposition 5.1.12, $1_G \cdot |f| m \in S_{00}$ (cf. Proposition 3.1.14). Thus the statement holds in the case of (i). The rest of the proof follows from (5.12) as in the proof of Lemma 3.2.6 (iii). □

We assume that for each $i, j = 1, \ldots, d$

(HP4) (i) if $\alpha \in (-d, -d+2]$, $\frac{\partial_j a_{ij}}{\rho} \in L^1_{loc}(\mathbb{R}^d, m) \cap L^q_{loc}(\mathbb{R}^d, dx)$, $0 < 2 - \frac{d}{q} < 1$,

(ii) if $\alpha \in (-d+2, 0)$, $\partial_j a_{ij} \in L^p_{loc}(\mathbb{R}^d, m)$ with $0 < 2 - \alpha - \frac{d}{p} < 1$ and $\frac{\partial_j a_{ij}}{\rho} \in L^q_{loc}(\mathbb{R}^d, dx)$ with $0 < 2 - \frac{d}{q} < 1$,

(iii) if $\alpha \in [0, 2)$, $\frac{\partial_j a_{ij}}{\rho} \in L^p_{loc}(\mathbb{R}^d, m)$ with $0 < 2 - \alpha - \frac{d}{p} < 1$.

Lemma 5.2.7. *Let ρ be as in* (5.11) *and $G \subset \mathbb{R}^d$ any relatively compact open set. Assume (HP4). Then for each $i, j = 1, \ldots, d$*

$$1_G \cdot \frac{a_{ii}}{\rho} m \in S_{00}, \quad 1_G \cdot \frac{|\partial_j a_{ij}|}{\rho} m \in S_{00}.$$

Proof. For any relatively compact open set $G \subset \mathbb{R}^d$ $1_G \cdot \frac{a_{ii}}{\rho} m$ and $1_G \cdot \frac{|\partial_j a_{ij}|}{\rho} m$ are positive finite measures on $\mathbb{R}^d$. Furthermore by (5.3), $1_G \cdot \frac{a_{ii}}{\rho} \in \mathcal{B}_b(\mathbb{R}^d)$. Therefore, by Proposition 5.1.10 (i) $R_1\Big(1_G \cdot \frac{a_{ii}}{\rho}\Big) \in C_b(\mathbb{R}^d)$. Since

$$R_1\Big(1_G \cdot \, \frac{a_{ii}}{\rho} m\Big)(x) = R_1\Big(1_G \cdot \frac{a_{ii}}{\rho}\Big)(x),$$

$1_G \cdot \frac{a_{ij}}{\rho} m \in S_{00}$ (see Proposition 5.1.12). By the assumption (HP4) and Lemma 5.2.6, $1_G \cdot \frac{|\partial_j a_{ij}|}{\rho} m \in S_{00}$. □

We will refer to [35] till the end, hence some of its standard notations may be adopted below without definition. Let $f^i(x) := x_i$, $i = 1, \ldots, d$, $x \in \mathbb{R}^d$, be the coordinate functions. Then for $f^i \in D(\mathcal{E}^A)_{b,loc}$ and any $g \in C_0^\infty(\mathbb{R}^d)$ the following integration by parts formula holds:

$$- \mathcal{E}^A(f^i, g) = \frac{1}{2} \sum_{j=1}^{d} \int_{\mathbb{R}^d} \frac{\partial_j a_{ij}}{\rho} g \, dm, \quad 1 \leq i \leq d. \tag{5.13}$$

Theorem 5.2.8. *Assume (HP1), (HP3),* (5.11)*, and (HP4). Then it holds $\mathbb{P}_x$-a.s. for any $x \in \mathbb{R}^d$, $i = 1, \ldots, d$*

$$X_t^i = x^i + \sum_{j=1}^{d} \int_0^t \frac{\sigma_{ij}}{\sqrt{\rho}}(X_s) \, dW_s^j + \frac{1}{2} \sum_{j=1}^{d} \int_0^t \frac{\partial_j a_{ij}}{\rho}(X_s) \, ds, \quad t \geq 0, \tag{5.14}$$

where $(\sigma_{ij})_{1 \leq i,j \leq d} = \sqrt{A}$ is the positive square root of the matrix A, $W = (W^1, \ldots, W^d)$ is a standard d-dimensional Brownian motion on $\mathbb{R}^d$.

Proof. By Lemma 5.2.7, (5.13), and [35, Theorem 5.5.5] the strict continuous additive functional, locally of zero energy and corresponding to the coordinate function $f^i \in D(\mathcal{E}^A)_{b,loc}$ is given by

$$N_t^{[f^i]} = \frac{1}{2} \sum_{j=1}^{d} \int_0^t \frac{\partial_j a_{ij}}{\rho}(X_s) \, ds, \quad t \geq 0, \quad 1 \leq i \leq d.$$

The energy measure of f^i denoted by $\mu_{\langle f^i \rangle}$ satisfies $\mu_{\langle f^i \rangle} = \frac{a_{ii}}{\rho} m$. Therefore by Lemma 5.2.7 for any relatively compact open set $G \subset \mathbb{R}^d$, $1_G \cdot \mu_{\langle f^i \rangle} \in S_{00}$ and so the positive continuous additive functional in the strict sense corresponding to the Reuvz measure $\mu_{\langle f^i \rangle}$ is given by

$$\langle M^{[f^i]} \rangle_t = \int_0^t \frac{a_{ii}}{\rho}(X_s) \, ds,$$

where $M_t^{[f^i]}$ is the continuous local martingale additive functional in the strict sense corresponding to f^i. Furthermore since the covariation is

$$\langle M^{[f^i]}, M^{[f^j]} \rangle_t = \int_0^t \frac{a_{ij}}{\rho}(X_s) \, ds,$$

we can construct a d-dimensional Brownian motion W (on a possibly enlarged probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}_x)$ that we call again w.l.o.g. $(\Omega, \mathcal{F}, \mathbb{P}_x)$) such that

$$M_t^{[f^i]} = \sum_{j=1}^{d} \int_0^t \frac{\sigma_{ij}}{\sqrt{\rho}}(X_s)\ dW_s^j,$$

where $(\sigma_{ij})_{1\le i,j\le d} = \sqrt{A}$ is the positive square root of the matrix A (cf. [42, Chapter 3, Theorem 4.2]). Note that the equation (5.14) holds for all $t \ge 0$ because $(\mathcal{E}^A, D(\mathcal{E}^A))$ is conservative (see Remark 5.1.11). □

5.2.2 Concrete weights with polynomial growth induced by quasi-conformal mappings

We consider

$$\rho(x) = \|x\|^{\alpha}, \quad \alpha \in [2, \infty), \quad d \ge 2. \tag{5.15}$$

Let

$$B_k := \{x \in \mathbb{R}^d \mid k^{-1} < \|x\| < k\}, \quad k \ge 1, \tag{5.16}$$

and for any $G \subset \mathbb{R}^d$

$$C^\infty(\overline{G}) := \{f : \overline{G} \to \mathbb{R} \mid \exists g \in C_0^\infty(\mathbb{R}^d), g|_{\overline{G}} = f\}.$$

According to [68] and (5.3) the closure of

$$\mathcal{E}^{A,\overline{B}_k}(f,g) := \frac{1}{2}\int_{B_k} \langle A\nabla f, \nabla g\rangle\ dx, \quad f,g \in C^\infty(\overline{B}_k),$$

in $L^2(\overline{B}_k, m) \equiv L^2(B_k, m)$, $k \ge 1$, denoted by $(\mathcal{E}^{A,\overline{B}_k}, D(\mathcal{E}^{A,\overline{B}_k}))$, is a regular Dirichlet form on $\overline{B}_k$ and moreover, it holds:

Lemma 5.2.9. *Let ρ be as in* (5.15).

(i) (Nash type inequality)

(a) If $d \ge 3$, then for $f \in D(\mathcal{E}^{A,\overline{B}_k})$

$$\|f\|_{2,B_k}^{2+\frac{4}{d}} \le c_k \left[\mathcal{E}^{A,\overline{B}_k}(f,f) + \|f\|_{2,B_k}^2\right] \|f\|_{1,B_k}^{\frac{4}{d}}.$$

(b) If $d = 2$, then for $f \in D(\mathcal{E}^{A,\overline{B}_k})$ and any $\delta > 0$

$$\|f\|_{2,B_k}^{2+\frac{4}{d+\delta}} \leq c_k \left[\mathcal{E}^{A,\overline{B}_k}(f,f) + \|f\|_{2,B_k}^2\right] \|f\|_{1,B_k}^{\frac{4}{d+\delta}}.$$

Here $c_k > 0$ is a constant which goes to infinity as $k \to \infty$.

(ii) We have for m-a.e. $x, y \in B_k$

(a) if $d \geq 3$, then

$$r_1^{B_k}(x,y) \leq c_1 \frac{1}{\|x-y\|^{d-2}}.$$

(b) if $d = 2$, then for any $\delta > 0$

$$r_1^{B_k}(x,y) \leq c_1 \frac{1}{\|x-y\|^{d+\delta-2}}.$$

Here $c_1 > 0$ is some constant.

Proof. The Nash type inequality (i) follows from (5.3), (5.16), and the proof of Lemma 3.4.4. Following the proof of Proposition 3.4.5, Corollary 3.4.6 the assertion (ii) follows. □

We assume that

(HP4)′ $\partial_j a_{ij} \in L_{loc}^{\frac{d}{2}+\varepsilon}(\mathbb{R}^d, dx)$ for some $\varepsilon > 0$ and each $i, j = 1, \ldots, d$.

Lemma 5.2.10. *Assume (HP4)′. Let ρ be as in* (5.15) *and $f \in L^{\frac{d}{2}+\varepsilon}(B_k, dx)$ for some $\varepsilon > 0$. Then*

$$1_{B_k} \cdot |f| m \in S_{00}^{B_k}.$$

In particular

$$1_{B_k} \cdot \frac{a_{ii}}{\rho} m \in S_{00}^{B_k}, \quad 1_{B_k} \cdot \frac{|\partial_j a_{ij}|}{\rho} m \in S_{00}^{B_k}.$$

Proof. Using Lemma 5.2.9 (ii) and (HP4)′ the proof is similar to Lemma 5.2.7, so we omit it (cf. Lemma 3.4.8). □

The following integration by parts holds true for the coordinate functions $f^i \in D(\mathcal{E}^{A,B_k})_{b,loc}$, $i = 1, \ldots, d$ and $g \in C_0^\infty(B_k)$:

$$-\mathcal{E}^{A,B_k}(f^i, g) = \frac{1}{2} \sum_{j=1}^{d} \int_{B_k} \frac{\partial_j a_{ij}}{\rho} g \, dm. \tag{5.17}$$

Proposition 5.2.11. *Assume (HP1), (HP3),* (5.15)*, and (HP4)$'$. Then the process* $\mathbb{M}$ *satisfies*

$$X_t^i = x^i + \sum_{j=1}^d \int_0^t \frac{\sigma_{ij}}{\sqrt{\rho}}(X_s)\, dW_s^j + \frac{1}{2}\sum_{j=1}^d \int_0^t \frac{\partial_j a_{ij}}{\rho}(X_s)\, ds, \quad t < D_{B_k^c}, \tag{5.18}$$

$\mathbb{P}_x$-a.s. for any $x \in B_k$, $i = 1, \dots, d$ where W is a standard d-dimensional Brownian motion starting from zero.

Proof. We apply [35, Theorem 5.5.5] to $(\mathcal{E}^{A,B_k}, D(\mathcal{E}^{A,B_k}))$. The assertion follows from Lemma 5.2.10 and (5.9), (5.17) (see Theorem 5.2.8 for details). □

Lemma 5.2.12. *Let $\alpha \in [-d+2, \infty)$. Then:*

(i) $\mathrm{Cap}(\{0\}) = 0$.

(ii) For all $x \in \mathbb{R}^d \setminus \{0\}$

$$\mathbb{P}_x\Big(\lim_{k\to\infty} D_{B_k^c} = \infty\Big) = \mathbb{P}_x\Big(\lim_{k\to\infty} \sigma_{B_k^c} = \infty\Big) = 1.$$

Proof. (i) By [35, Example 3.3.2] and (5.3), $\mathrm{Cap}(\{0\}) = 0$ if $\alpha \in [-d+2, \infty)$. (ii) This follows from (i), (5.16), Theorem 5.1.8, and Lemma 3.4.10. □

Theorem 5.2.13. *Assume (HP1), (HP3),* (5.15)*, and (HP4)$'$. Then the process* $\mathbb{M}$ *satisfies* (5.14) *for all $x \in \mathbb{R}^d \setminus \{0\}$.*

Proof. By Lemma 5.2.12, we can let k go to infinity in (5.18). □

Remark 5.2.14. *In Proposition 3.2.8 we considered the Muckenhoupt A_2-weight $\rho(x) = \|x\|^\alpha$, $\alpha \in (-d+1, d)$, the conclusion of which can be immediately derived from our results by setting $A = \rho{\cdot}Id$. Moreover, we can extend the restriction on $\alpha \in (-d+1, d)$ to $\alpha \in (-d+1, \infty)$ as shown in this section.*

5.3 Muckenhoupt A_2-weights with exponential growth

Let $\phi \in C(\mathbb{R}^d) \cap L^1_{loc}(\mathbb{R}^d, dx)$ such that for every cube $Q \subset \mathbb{R}^d$, $d \geq 2$

$$\int_Q e^{|\phi(x)-\phi_Q|}\, dx \leq c \int_Q 1\, dx,$$

where c is a constant independent of the cube Q and $\phi_Q = \frac{1}{dx(Q)} \int_Q \phi\, dx$.
We consider

$$\rho(x) := e^{\phi(x)}, \quad x \in \mathbb{R}^d. \tag{5.19}$$

Then by [37, IV. Corollary 2.18] $\rho \in A_2$ and ρ satisfies (HP2).

In Section 3.1 we considered a symmetric, strongly local, regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E,m)$ admitting carré du champ where E is a locally compact separable metric space and m is a positive Radon measure on $(E, \mathcal{B}(E))$ with full support on E.

There we assumed

(**H1**) There exists a $\mathcal{B}(E) \times \mathcal{B}(E)$ measurable non-negative map $p_t(x,y)$ such that

$$P_t f(x) := \int_E p_t(x,y)\, f(y)\, m(dy)\,,\ t>0, \quad x \in E, \quad f \in \mathcal{B}_b(E),$$

is a (temporally homogeneous) sub-Markovian transition function (see [22, 1.2]) and an m-version of $T_t f$ if $f \in L^2(E,m)_b$.

(**H2**)$'$ We can find $\{u_n \mid n \geq 1\} \subset D(L) \cap C_0(E)$ satisfying:

- (i) For all $\varepsilon \in \mathbb{Q} \cap (0,1)$ and $y \in D$, where D is any given countable dense set in E, there exists $n \in \mathbb{N}$ such that $u_n(z) \geq 1$, for all $z \in \overline{B}_{\frac{\varepsilon}{4}}(y)$ and $u_n \equiv 0$ on $E \setminus B_{\frac{\varepsilon}{2}}(y)$.
- (ii) $R_1\big([(1-L)u_n]^+\big)$, $R_1\big([(1-L)u_n]^-\big)$, $R_1\big([(1-L_1)u_n^2]^+\big)$, $R_1\big([(1-L_1)u_n^2]^-\big)$ are continuous on E for all $n \geq 1$.
- (iii) $R_1 C_0(E) \subset C(E)$.
- (iv) For any $f \in C_0(E)$ and $x \in E$, the map $t \mapsto P_t f(x)$ is right-continuous on $(0,\infty)$.

Under (**H1**) and (**H2**)$'$ we showed that there exists a Hunt process with transition function $(P_t)_{t\geq 0}$ (see Lemma 3.1.10). We intend to do the same here for our concrete example, i.e. we will derive conditions on a_{ij} that imply (**H1**) and (**H2**)$'$.

We hence assume in this section that

(HP5) For $i, j = 1, \ldots, d$
$$\partial_j a_{ij} \in L^\infty_{loc}(\mathbb{R}^d, dx).$$

Note that by (5.3) $\sum_{i,j=1}^d \frac{a_{ij}}{\rho} \leq d \cdot \lambda$ and by (HP5) $\frac{\partial_j a_{ij}}{\rho} \in L^\infty_{loc}(\mathbb{R}^d, dx)$, $i, j = 1, \ldots, d$. Therefore for $f \in C_0^\infty(\mathbb{R}^d) \subset D(L)$

$$Lf = -\sum_{i,j=1}^d \left(\frac{a_{ij}}{\rho}\, \partial_{ij} f + \frac{\partial_j a_{ij}}{\rho}\, \partial_i f \right) \in L^\infty(\mathbb{R}^d, m)_0. \tag{5.20}$$

Theorem 5.3.1. *Assume (HP5). There exists a Hunt process $\mathbb{M}$ satisfying the absolute continuity condition.*

Proof. By Proposition 3.2.3 (ii) (**H1**) and **(H2)$'$** (iii), (iv) hold true with the help of transition density estimate replaced by (5.7). Clearly we can find $\{u_n \mid n \geq 1\} \subset C_0^\infty(\mathbb{R}^d) \subset D(L)$ such that (**H2**)$'$ (i) is satisfied. Furthermore (**H2**)$'$ (ii) follows from (5.20) and Proposition 5.1.10 (i). □

Let
$$D_k := \{x \in \mathbb{R}^d \mid \|x\| < k\}, \quad k \geq 1.$$

Note that the ρ is bounded below and above on each D_k, $k \geq 1$. Then using Nash type inequality as in Lemma 5.2.9 (i) with $\overline{B}_k$ replaced by $\overline{D}_k$ we can obtain for m-a.e. $x, y \in D_k$ resolvent density estimates

$$r_1^{D_k}(x, y) \leq c_1 \frac{1}{\|x - y\|^{d-2}}, \quad \text{if } d \geq 3, \tag{5.21}$$

and for any $\delta > 0$
$$r_1^{D_k}(x, y) \leq c_1 \frac{1}{\|x - y\|^{d+\delta-2}}, \quad \text{if } d = 2, \tag{5.22}$$

where c_1 is some constant.

Lemma 5.3.2. *Assume (HP5) and $d \geq 2$. Then:*

$$1_{D_k} \cdot \frac{a_{ii}}{\rho} m \in S_{00}^{D_k}, \quad 1_{D_k} \cdot \frac{|\partial_j a_{ij}|}{\rho} m \in S_{00}^{D_k}.$$

Proof. Using the resolvent density estimates (5.21), (5.22) we can show this similarly to the proof of Lemma 5.2.10. □

Note that the integration by parts (5.17) holds true for B_k replaced by D_k. Following the proof of Theorem 5.2.13 we obtain:

Theorem 5.3.3. *Assume (HP1), (HP3),* (5.19)*, and (HP5). The process* $\mathbb{M}$ *in Theorem 5.3.1 satisfies* (5.14) $\mathbb{P}_x$*-a.s. for any* $x \in \mathbb{R}^d$*.*

5.4 Pathwise uniqueness and strong solution

In this section we assume

(HP6) For each $1 \le i, j \le d$,

(i) $\frac{\sigma_{ij}}{\sqrt{\rho}}$ is continuous on $\mathbb{R}^d$.

(ii) $\nabla \left(\frac{\sigma_{ij}}{\sqrt{\rho}} \right) \in L^{2(d+1)}_{loc}(\mathbb{R}^d, dx)$.

(iii) $\sum_{k=1}^{d} \frac{\partial_k a_{ik}}{\rho} \in L^{2(d+1)}_{loc}(\mathbb{R}^d, dx)$.

Theorem 5.4.1. *Assume that (HP1), (HP3),* (5.11)*, (HP4), and (HP6) hold true. Then for any* $x \in \mathbb{R}^d$ *the (weak) solution in Theorem 5.2.8 is strong and pathwise unique. In particular, it is adapted to the filtration* $(\mathcal{F}_t^W)_{t \ge 0}$ *generated by the Brownian motion* $(W_t)_{t \ge 0}$ *as in* (5.14)*.*

Proof. By [73, Theorem 1.1] under (HP6) for given Brownian motion $(W_t)_{t \ge 0}$, $x \in \mathbb{R}^d$ as in (5.14) there exists a pathwise unique strong solution to (5.14). Therefore for any $x \in \mathbb{R}^d$ the (weak) solution in Theorem 5.2.8 is strong and pathwise unique. □

Remark 5.4.2. *(i) If we additionally assume (HP6) in Theorem 5.3.3, the (weak) solution in Theorem 5.3.3 is also strong and pathwise unique (see Theorem 5.4.1). However we do not know whether the solution in Theorem 5.2.13 with additional assumption (HP6) is strong or not because [73, Theorem 1.1] can be applied only when the state space is* $\mathbb{R}^d$*.*

(ii) Two non-explosion conditions for the strong solution in Theorem 5.4.1 are presented in [73, Theorem 1.1]. By Theorem 5.2.8 and 5.4.1 we know that the strong solution is associated to the Dirichlet form $(\mathcal{E}^A, D(\mathcal{E}^A))$*. Hence Dirichlet form theory can provide non-explosion criterion for the solution, i.e. conservativeness*

of the Dirichlet form $(\mathcal{E}^A, D(\mathcal{E}^A))$ gives another non-explosion criterion (cf. Remark 5.1.11).

Appendix A

Appendix

This chapter consists of auxiliary results of Chapter 4. First we present an alternative proof of Lemma 4.2.4, which does not require the absolute continuity condition.

Lemma A.0.3. *Let $(B_k)_{k\geq 1}$ be an increasing sequence of relatively compact open sets in E with $\cup_{k\geq 1} B_k = E$. Then for all $x \in E$*

$$\mathbb{P}_x\Big(\lim_{k\to\infty} \sigma_{E\setminus B_k} \geq \zeta\Big) = 1.$$

Proof. Let $(B_k)_{k\geq 1}$ be an increasing sequence of relatively compact open sets in E with $\cup_{k\geq 1} B_k = E$ and $\sigma := \lim_{k\to\infty} \sigma_{E\setminus B_k}$. By quasi-left-continuity of $\mathbb{M}$

$$\mathbb{P}_x\Big(\lim_{k\to\infty} X_{\sigma_{E\setminus B_k}} = X_\sigma,\ \sigma < \infty\Big) = \mathbb{P}_x(\sigma < \infty), \quad \forall x \in E. \tag{A.1}$$

Using [35, Lemma A.2.7], it follows for any $k \geq 1$ that

$$\mathbb{P}_x\Big(X_{\sigma_{E\setminus B_k}} \in (E \setminus B_k) \cup \{\Delta\},\ \sigma_{E\setminus B_k} < \infty\Big) = \mathbb{P}_x(\sigma_{E\setminus B_k} < \infty), \quad \forall x \in E,$$

hence

$$\mathbb{P}_x\Big(X_{\sigma_{E\setminus B_k}} \in (E \setminus B_k) \cup \{\Delta\},\ \sigma < \infty\Big) = \mathbb{P}_x(\sigma < \infty), \quad \forall x \in E. \tag{A.2}$$

From (A.1) and (A.2)

$$\mathbb{P}_x\Big(\lim_{k\to\infty} X_{\sigma_{E\setminus B_k}} = X_\sigma,\ X_{\sigma_{E\setminus B_k}} \in (E\setminus B_k)\cup\{\Delta\},\ \forall k \geq 1,\ \sigma < \infty\Big) = \mathbb{P}_x(\sigma < \infty), \quad \forall x \in E.$$

Let

$$
\begin{aligned}
A &:= \Big\{ \lim_{k\to\infty} X_{\sigma_{E\setminus B_k}} = X_\sigma,\ X_{\sigma_{E\setminus B_k}} \in (E \setminus B_k) \cup \{\Delta\},\ \forall k \geq 1,\ \sigma < \infty \Big\}, \\
B &:= \Big\{ X_\sigma \in \{\Delta\} \Big\}.
\end{aligned}
$$

Suppose, to show $A \subset B$, that $\omega \in A$ but $\omega \notin B$, i.e. there exists $x \in E$ such that $X_{\sigma(\omega)}(\omega) = x$ with $\omega \in A$. Since E is open in $\mathbb{R}^d$, we can find a ball $B_\varepsilon(x)$, $\varepsilon > 0$ such that the closure $\overline{B_\varepsilon(x)} \subset E$. Since $(B_k)_{k\geq 1}$ is an open cover of $\overline{B_\varepsilon(x)}$ and increasing, we can find $k^\star \in \mathbb{N}$ such that $B_k \supset \overline{B_\varepsilon(x)}$ for all $k \geq k^\star$. Since $\omega \in A$, this implies that $X_{\sigma_{E\setminus B_k}(\omega)}(\omega) \notin B_\varepsilon(x)$, $k \geq k^\star$ and so $\lim_{k\to\infty} X_{\sigma_{E\setminus B_k}(\omega)}(\omega) \notin B_\varepsilon(x)$, which draws a contradiction. Hence

$$
\mathbb{P}_x\Big(X_\sigma \in \{\Delta\},\ \sigma < \infty\Big) = \mathbb{P}_x(\sigma < \infty), \quad \forall x \in E,
$$

and so

$$
\mathbb{P}_x\Big(\sigma \geq \zeta,\ \sigma < \infty\Big) = \mathbb{P}_x(\sigma < \infty), \quad \forall x \in E.
$$

Clearly

$$
\mathbb{P}_x\Big(\sigma \geq \zeta,\ \sigma = \infty\Big) = \mathbb{P}_x(\sigma = \infty), \quad \forall x \in E,
$$

thus

$$
\mathbb{P}_x(\sigma \geq \zeta) = 1, \quad \forall x \in E.
$$

□

Assuming $\mathbb{M}$ is a diffusion process, we obtain the following shift-invariant result:

Lemma A.0.4. *Let $(B_k)_{k\geq 1}$ be an increasing sequence of sets in E. Then the set $\Big\{ \lim_{k\to\infty} \sigma_{E\setminus B_k} \geq \zeta \Big\}$ is shift-invariant.*

Proof. Let

$$
\Lambda := \Big\{ \lim_{k\to\infty} \sigma_{E\setminus B_k} \geq \zeta \Big\}, \quad \omega \in \Lambda, \quad s > 0.
$$

If $s \geq \zeta(\omega)$, then

$$
\lim_{k\to\infty} \sigma_{E\setminus B_k} \circ \theta_s(\omega) = \zeta \circ \theta_s(\omega) = 0.
$$

If $s < \zeta(\omega)$, then

$$
\lim_{k\to\infty} \sigma_{E\setminus B_k} \circ \theta_s(\omega) + s \geq \lim_{k\to\infty} \sigma_{E\setminus B_k}(\omega) \geq \zeta(\omega).
$$

Hence

$$\lim_{k\to\infty} \sigma_{E\setminus B_k} \circ \theta_s(\omega) \geq \zeta \circ \theta_s(\omega).$$

This implies that

$$\theta_s \circ \Lambda \subset \Lambda.$$

Conversely, for $\omega \in \Lambda$, let

$$\omega'(t) := \omega(0), \quad t \in [0, s) \quad \text{and} \quad \omega'(t) := \omega(t - s), \quad t \in [s, \infty).$$

Then $\omega' \in \Lambda$ and $\omega = \theta_s(\omega')$. Therefore

$$\Lambda \subset \theta_s \circ \Lambda.$$

$\square$

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국문초록

우선 Fukushima의 절대 연속 조건 가정 하에서 $\mathbb{R}^d$ 상에 있는 어느 시작점에 대해서도 우리는 비연속인 Muckenhoupt A_2 가중과 연관된 비틀어진 브라운 운동의 확률 미분방정식을 푼다. 그런 후 우리는 체계적으로 절대 연속 조건을 적용 시킬 수 있는 일반적인 도구들을 발전 시켜 나간다. 이러한 도구들은 국소적으로 컴팩트한 분해가능한 거리 공간에서 참고측도에 대한 밀도를 갖는 추이 함수의 헌트 과정을 얻는 방법들과 표류 포텐셜을 추정할 수 있는 방법들로 구성되어 있다. 이 결과들은 비틀어진 브라운 운동에 적용될 수 있고 특이 확률 미분방정식, 즉 유계하지 않은 그리고 비연속적인 표류와 반사 항들 (셀 수 있는 만큼의 국소 시간들의 합)을 가지고 있는 방정식, 에 대한 약 해를 건설할 수 있다. 이 해들은 명확히 구체화된 상태공간의 어떤 점에서도 출발할 수 있다. 우리는 Muckenhoupt A_2 가중과 다양한 종류의 여러 경계 조건들 뿐만 아니라 특이점들에서 적절한 증가를 가지는 가중 등 여러가지 종류의 가중들을 고려한다. 우리는 또한 이러한 방법들을 퇴화 타원 디리클레 형식에 적용하여 이에 대응되는 확률 미분방정식의 해를 보인다. 마지막으로 우리는 대칭 비틀어진 브라운 운동의 결과들을 비대칭인 경우로 확장시킨다. 가중치 있는 공간에서 타원형 정칙성 결과들과 확률 미적분학 그리고 비대칭 디리클레 형식 이론들을 사용해서 우리는 $\mathbb{R}^d$의 부분 공간 E의 어느 시작점에서도 비대칭 비틀어진 브라운 운동의 약 해를 보인다. 여기서 E는 불변측도에 대한 밀도의 유일한 연속 버전의 순 양의 점들로 구체적으로 주어진다. 약 존재성을 보인 후에 우리는 [43]의 결과로부터 건설된 약해가 실제로 폭발 시간까지 경로별 유일한 강해라는 것을 보인다. 우리의 접근 방식의 결과로서 해의 성질들을 규명하는데 디리클레 형식을 사용할 수 있다. 보다 구체적으로 우리는 해들에 대한 새로운 비폭발성 기준들을 얻는다.

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